K.RAMAKRISHNAN COLLEGE OF TECHNOLOGY
SAMAYAPURAM, TRICHY-621 112

EC8352-SIGNALS & SYSTEMS
COURSE MATERIAL

DEPARTMENT OF ELECTRONICS & COMMUNICATION ENGINEERING

UNIT-1

CLASSIFICATION OF SIGNALS & SYSTEMS

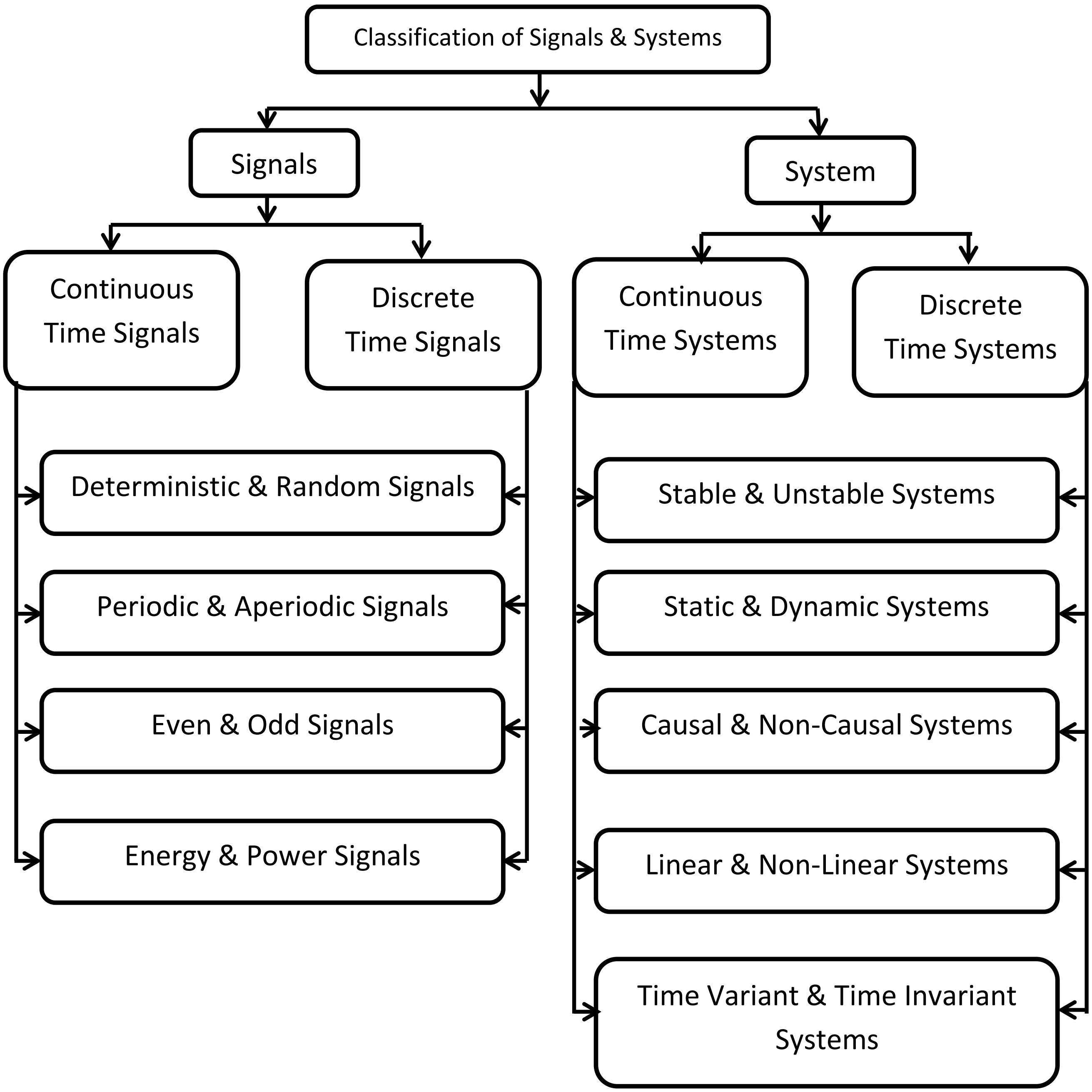
Syllabus:

Standard Signals-Step, Ramp, Pulse, Impulse, Real and Complex exponential and Sinusoidal Signals. Classification of Signals- Continuous Time (CT) and Discrete Time (DT) Signals, Periodic and Aperiodic Signals, Deterministic Signals & Random Signals, Energy & Power Signals-Classification of systems CT systems and DT systems- Linear & Nonlinear, Time Variant & Time Invariant, Causal & Non-Causal, Stable & Unstable Systems

Session	Session Learning	Topic
1	**1**	Signal and its representation
	2	Basic Elementary signals
2	**1**	Transformation of Continuous Time Signals
	2	Transformation of Discrete Time Signals
3	**1**	Operation of Continuous Time Signals
	2	Operation of Discrete Time Signals
4	**1**	Periodic and Aperiodic for Continuous Time Signals
	2	Periodic and Aperiodic for Discrete Time Signals
5	**1**	Even and Odd Signals for Continuous Time Signals
	2	Even and Odd Signals for Discrete Time Signals
6	**1**	Energy and Power Signals for Continuous Time Signals
	2	Energy and Power Signals for Continuous Signals
7	**1**	Energy and Power Signals for Discrete Time Signals
	2	Energy and Power Signals for Discrete Time Signals
8	**1**	System and its representation, Relation between Signals & Systems
	2	Stable & unstable Systems
9	**1**	Causal & Non Causal Systems
	2	Static and dynamic Systems
10	**1**	Linear and nonlinear for Continuous Time Systems
	2	Linear and nonlinear for Continuous Time Systems
11	**1**	Linear and nonlinear for Discrete Time Systems
	2	Linear and nonlinear for Discrete Time Systems
12	**1**	Time Variant and Time invariant for Continuous Time Systems
	2	Time Variant and Time invariant for Discrete Time Systems

Content

Flow Diagram



UNIT-1
CLASSIFICATION OF SIGNALS & SYSTEMS

Signal:

- It is a physical quantity which may vary with respect to independent variables, such as time, frequency, etc.
- In general signal is an information
- It is denoted as x(t).

Continuous Time Signal

- It is a signal in which the amplitude can be measured at any time instant
- It is denoted as x(t)

Discrete Time Signal

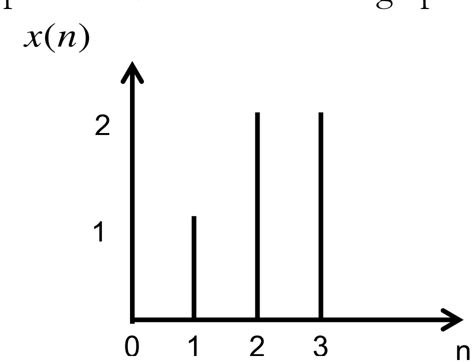
- It is a signal in which the amplitude can be measured at particular time instant
- It is denoted as x(n)

Representation of Signal:

- **Functional Representation**
  - It is represented in the form of function along with the time domain notation

$$x(n) = \begin{cases} 1; n = 0,1 \\ 2; n = 2,3 \\ 0; \quad else \end{cases}$$

- **Graphical Representation**
  - It is represented in the form of graphical structure



- **Tabular Representation**
  - It is represented in the form of tabular structure with the amplitude and the time specification

x(n)	1	1	2	2
n	0	1	2	3

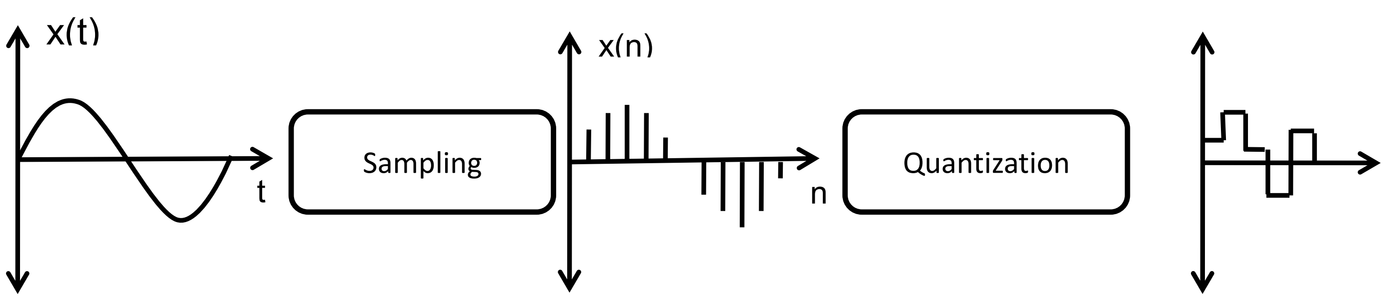
- **Sequence Representation**
  - It is represented in sequence form
  - In which arrow mark used to represent the origin position

$$x(n) = \{1, 1, 2, 2\}$$

Conversion of Analog Signal to Digital Signal

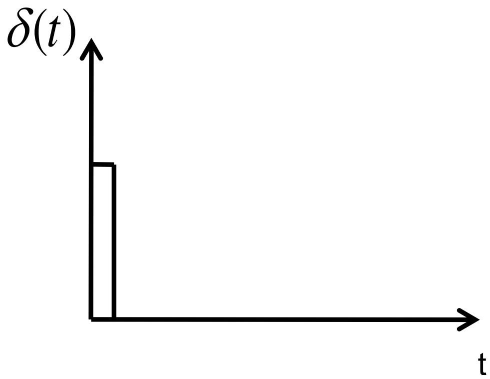
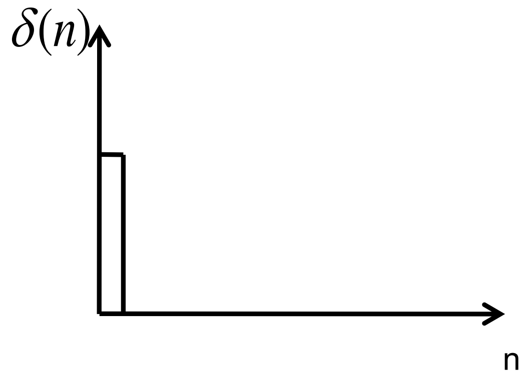
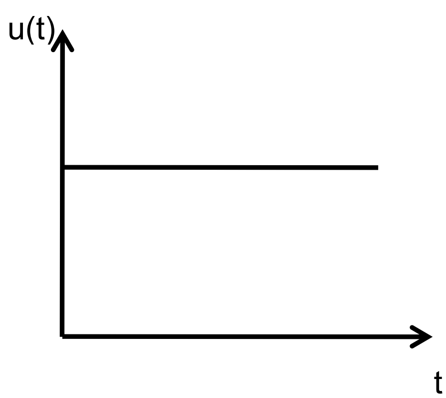
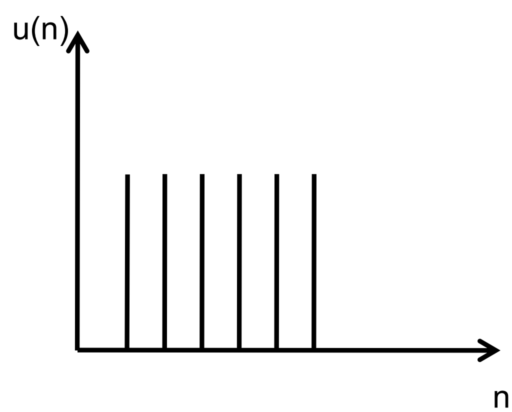
i) **Sampling: The process of converting the continuous time signal into discrete time signal**

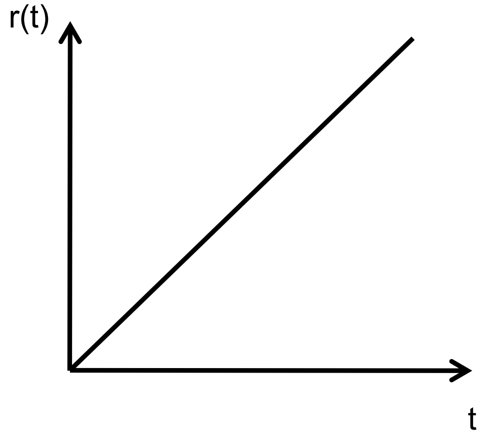
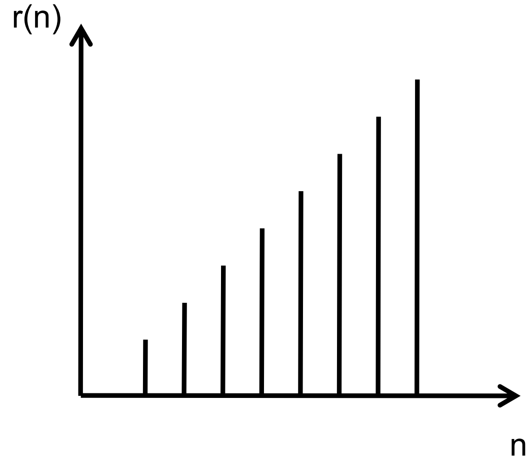
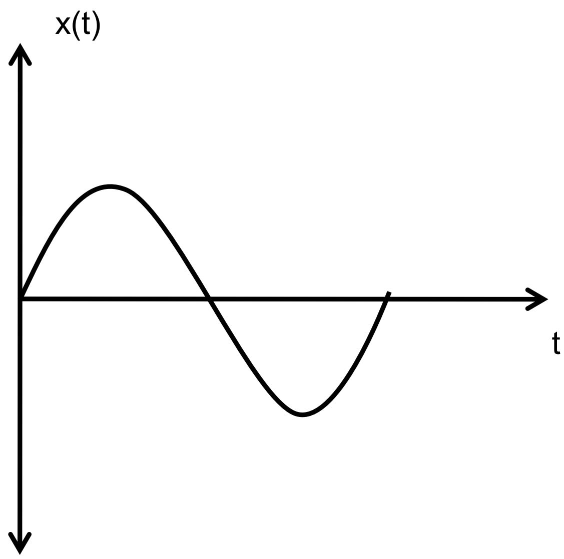
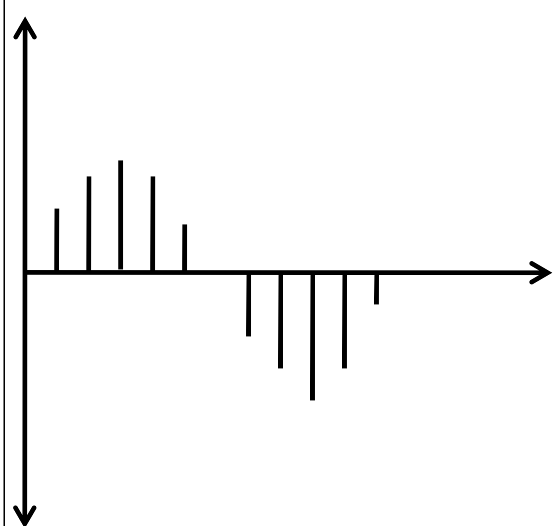
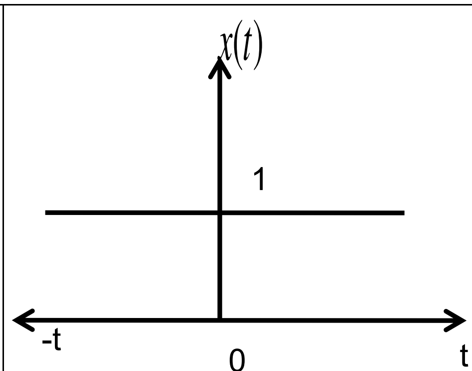
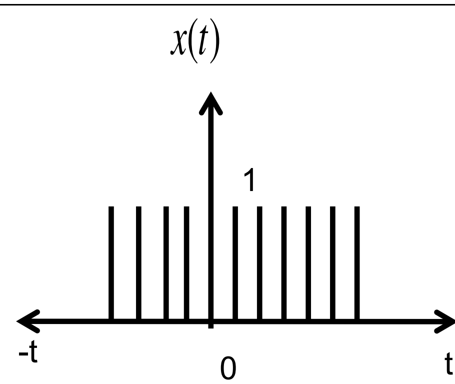
ii) **Quantization: The process of converting the discrete time signal into digital signal**

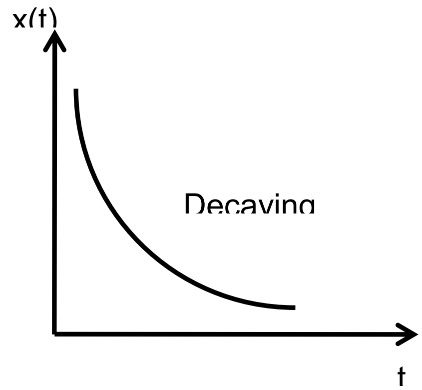
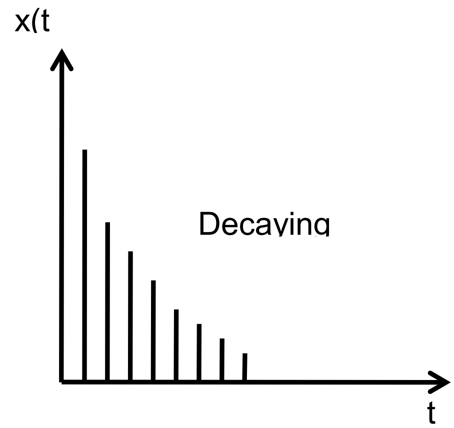
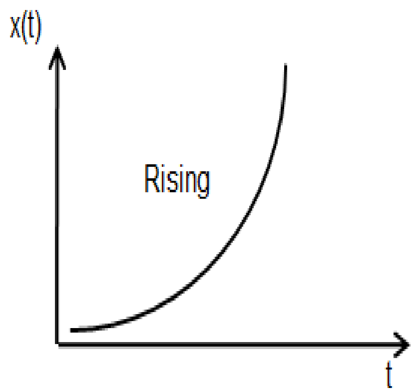
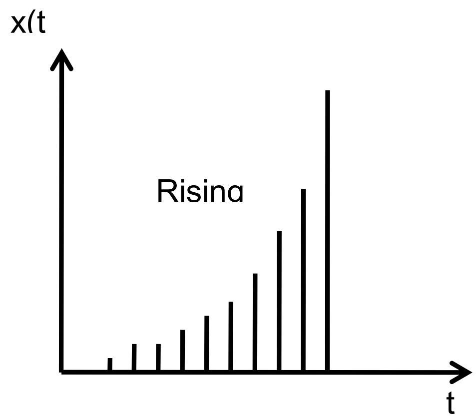
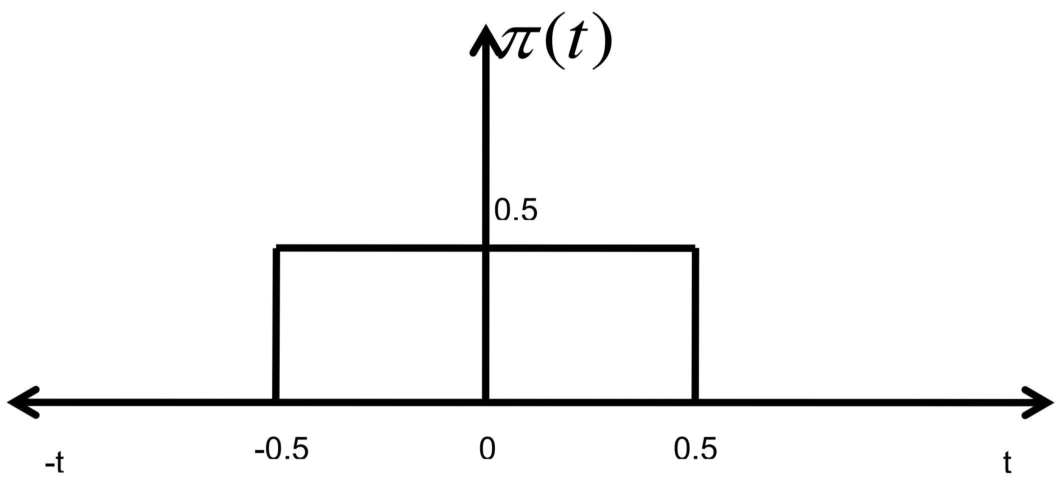


Basic Elementary Signals

It is a basic signals, which is used to test the system performance. So, it is also called as the Test Signals/Reference Signals

Signal	Continuous Time Signal	Discrete Time Signal
Unit Impulse Signal	$\delta(t) = \begin{cases} 1\ ; & t = 0 \\ 0\ ; & Otherwise \end{cases}$  	$\delta(n) = \begin{cases} 1\ ; & n = 0 \\ 0\ ; & Otherwise \end{cases}$ 
Unit Step Signal	$u(t) = \begin{cases} 1\ ;\ t \geq 0 \\ 0\ ;\ else \end{cases}$  	$u(n) = \begin{cases} 1\ ;\ n \geq 0 \\ 0\ ;\ else \end{cases}$ 
Unit Ramp Signal	$r(t) = \begin{cases} t\ ;\ t \geq 0 \\ 0\ ;\ else \end{cases}$	$r(n) = \begin{cases} n\ ;\ n \geq 0 \\ 0\ ;\ else \end{cases}$

		
Sinusoidal Signal	$x(t) = \sin \omega_o t$ 	$x(n) = \sin \omega_o n$ 
Exponential Signal	$a = 0$ 	

	$a<0$		
	$a>0$		
Rectangular Signal	$\pi(t)=\begin{cases} \frac{1}{2} \; ; & \lvert t\rvert \le \frac{1}{2} \\ 0 \; ; & Otherwise \end{cases}$ 		

Transformation of Signals

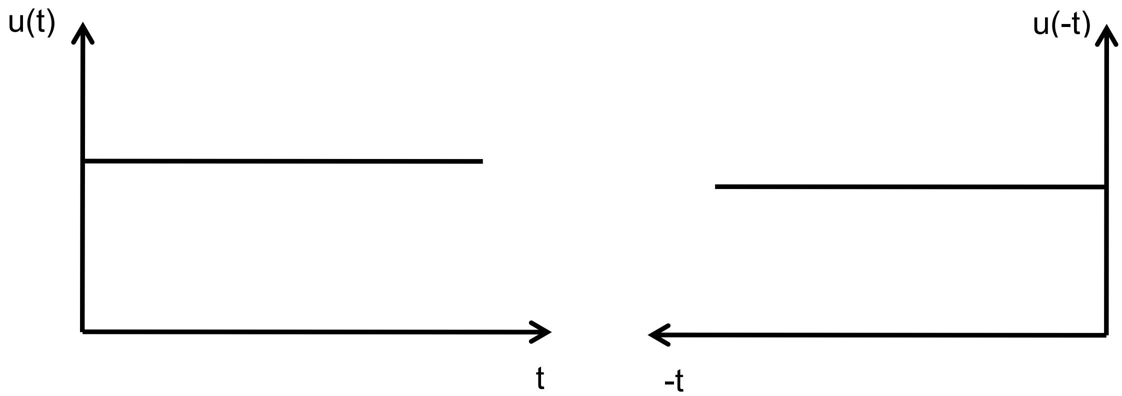
The changes occur in the time axis of the signal, then it is called as the transformation of signals.

- ❖ **Time Reversal :**

The folding of a continuous time signal X(t) is performed by changing the sign of time base t in the signal x(t).The folding operation produces a signal x(-t) which is a mirror image of the original signal x(t) with respect to the time origin t=0.

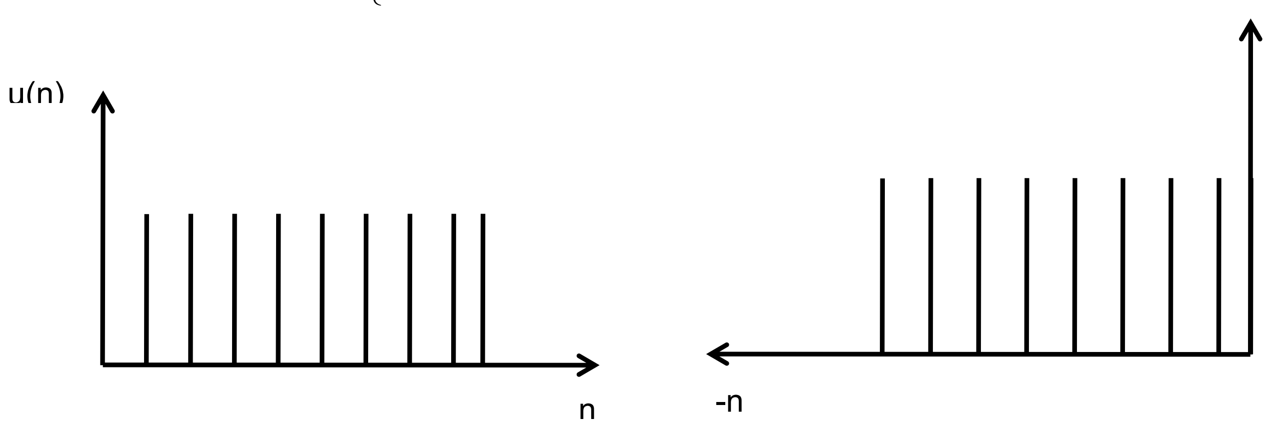
- Continuous Time Signal
  - $x(t) = u(-t)$

    $u(t) = \begin{cases} 1 \; ; \; t \geq 0 \\ 0 \; ; \; else \end{cases}$



- Discrete Time Signal
  - $x(n) = u(-n)$

    $u(n) = \begin{cases} 1 \; ; \; n \geq 0 \\ 0 \; ; \; else \end{cases}$



- Time Scaling :

  The time scaling is performed by multiplying the variable time by a constant.

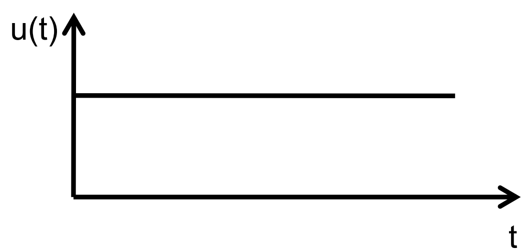
- Time Shifting :

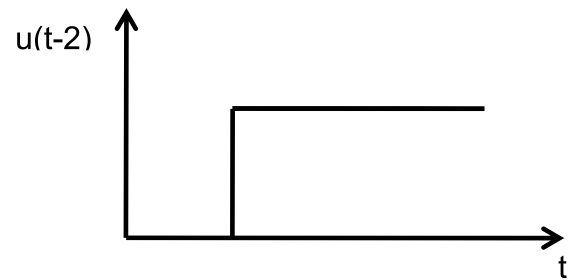
  The time shifting of a continuous time signal x(t) is performed by replacing the independent variable t by t-m, to get the time shifted signal x(t-m), where m represents the time shift in seconds.

  - Continuous Time Signal

1. $x(t) = u(t-2)$

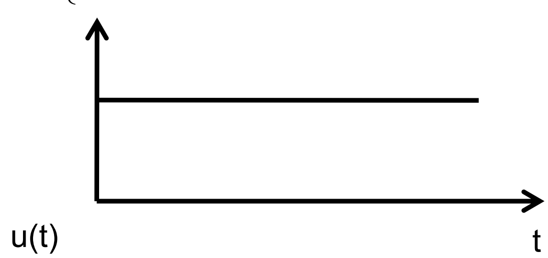
$$u(t) = \begin{cases} 1\ ;\ t \geq 0 \\ 0\ ;\ else \end{cases}$$

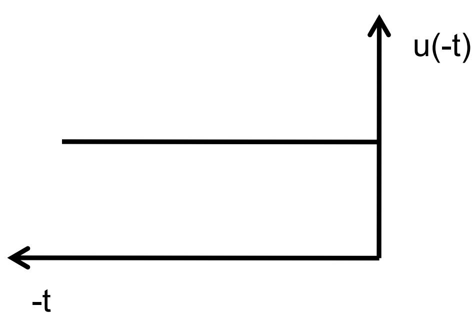


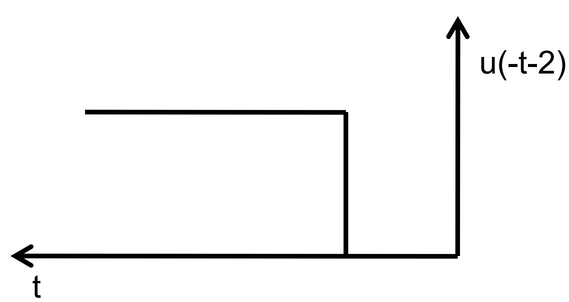


2. $x(t) = u(-t-2)$

$$u(t) = \begin{cases} 1\ ;\ t \geq 0 \\ 0\ ;\ else \end{cases}$$

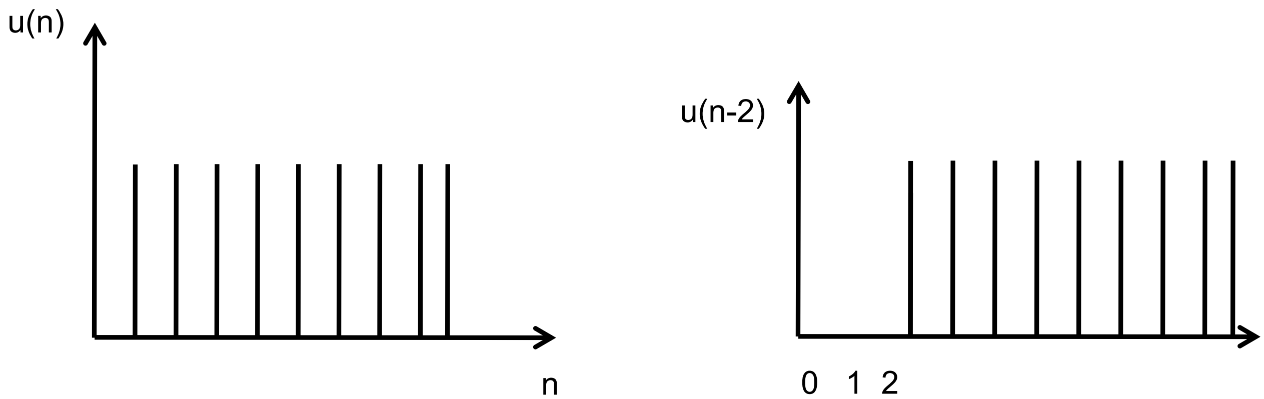




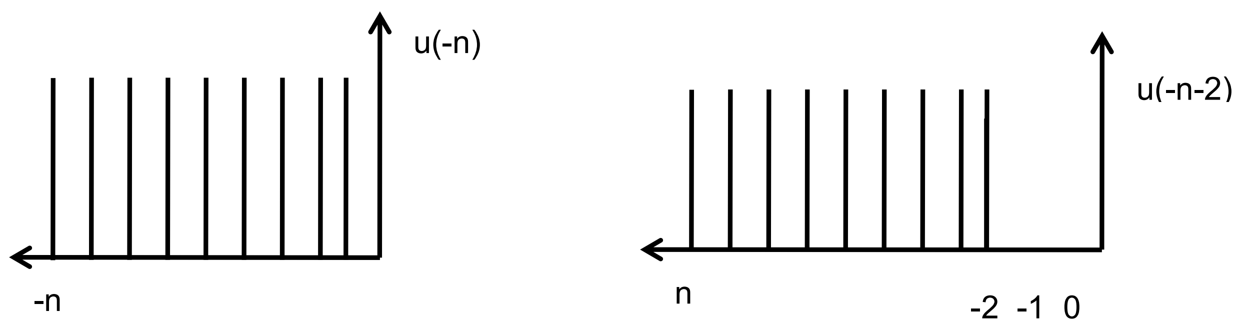


- Discrete Time Signal

1. $x(n) = u(n-2)$



2. $x(n) = u(-n-2)$



Operation of Signals

The changes occur in the amplitude of the signal, then it is called as the operation of signals.

- Addition of Signals: Addition of two signals
  - The addition of two continuous time signals is performed by adding the value of the two signals corresponding to the same instant of time.
  - For Continuous time Signal: $y(t) = x_1(t) + x_2(t)$
  - For Discrete Time Signal: $y(n) = x_1(n) + x_2(n)$

- Subtraction of Signals: Subtraction of two signals
  - The addition of two continuous time signals is performed by adding the value of the two signals corresponding to the same instant of time.
  - For Continuous time Signal: $y(t) = x_1(t) - x_2(t)$
  - For Discrete Time Signal: $y(n) = x_1(n) - x_2(n)$

- Multiplication of Signals: Multiplication of two signals

- The multiplication of two continuous time signals is performed by multiplying the value of the two signals corresponding to the same instant of time.
- For Continuous time Signal: $y(t) = x_1(t)x_2(t)$
- For Discrete Time Signal: $y(n) = x_1(n)x_2(n)$

Classification of signals

The continuous time signals are classified depending on their characteristics. Some ways of classifying continuous time signals are,

1. Deterministic and Nondeterministic (Random) signals
2. Periodic and Non periodic signals
3. Symmetric and Anti symmetric signals(Even and Odd signals)
4. Energy and Power signals

1. Deterministic and Nondeterministic (Random) signals

Deterministic Signals	**Random Signals**
• The signal that can be completely specified by a mathematical equation • The amplitude can be measured (or) Characteristics is known • Eg.: Step, Ramp, Exponential Signal	• The signal whose characteristics are random in nature • The amplitude cannot be measured • Eg.: noise signals from various sources like electronics amplifiers, oscillators, radio receivers

2. Periodic and Non periodic signal

Periodic Signals	Aperiodic Signals
➢ Periodic signal will have a definite pattern that repeats again and again over a certain period of time. ➢ Satisfies $x(t) = x(t+T)$; $x(n) = x(n+N)$	➢ The same information is not repeated again and again over a certain period of time. ➢ Satisfies $x(t) \neq x(t+T)$; $x(n) \neq x(n+N)$

In periodic signals, the term

T is called the fundamental time period of the signal.

$\mathbf{F_o}$ Hence inverse of T is called the fundamental frequency(Hz)

$2\pi F_0 = \mathbf{\Omega}_0$ is called the fundamental angular frequency (rad/sec)

Proof for Periodic Signals

a) Co-sinusoidal signal

$$Let,\ X(t) = A\cos\Omega_0 t$$

$$\therefore X(t+T) = A\cos\Omega_0(t+T) = A\cos(\Omega_0 t + \Omega_0 T)$$

$$= A\cos(\Omega_0 t + \frac{2\pi}{T}T)$$

$$= A\cos(\Omega_0 t + 2\pi) = A\cos\Omega_0 t = X(t)$$

b) Sinusoidal signal

$$Let,\ X(t) = A\sin\Omega_0 t$$

$$\therefore X(t+T) = A\sin\Omega_0(t+T) = A\sin(\Omega_0 t + \Omega_0 T)$$

$$= A\sin(\Omega_0 t + \frac{2\pi}{T}T)$$

$$= A\sin(\Omega_0 t + 2\pi) = A\sin\Omega_0 t = X(t)$$

c) Complex exponential signal

$$Let,\ X(t) = Ae^{j\Omega_0 t}$$

$$\therefore X(t+T) = Ae^{j\Omega_0(t+T)} = Ae^{j\Omega_0 t}e^{j\Omega_0 T}$$

$$= Ae^{j\Omega_0 t}e^{j\frac{2\pi}{T}T} = Ae^{j\Omega_0 t}e^{j2\pi}$$

$$= Ae^{j\Omega_0 t}(\cos 2\pi + j\sin 2\pi)] = Ae^{j\Omega_0 t}(1 + j0) = X(t)$$

1. Check whether the given signal is periodic or aperiodic

   a. $x(t) = \cos 40\pi t$

      $Fundamental\ \ Frequency\ \Omega_o = 40\pi$

      $\Omega_o T = 2\pi$

      $T = \frac{2\pi}{\Omega_o} = \frac{2\pi}{40\pi} = \frac{1}{20}$

      $Fundamental\ \ Period\, is\ \ rational \Rightarrow Periodic\ \ Signal$

   b. $x(t) = \cos(40\pi t + 1) + \sin(30t + 1)$

      $Fundamental\ \ Frequency\ \Omega_{o1} = 40\pi$

      $Fundamental\ \ Frequency\ \Omega_{o2} = 30$

      $\Omega_o T = 2\pi$

      $T_1 = \frac{2\pi}{\Omega_{o1}} = \frac{2\pi}{40\pi} = \frac{1}{20}$

      $T_2 = \frac{2\pi}{\Omega_{o2}} = \frac{2\pi}{30} = \frac{\pi}{15}$

      $T = \frac{T_1}{T_2} = \frac{1/20}{\pi/15} = \frac{15}{20\pi} = \frac{3}{4\pi}$

      $Fundamental\ \ Period\, is\ \ irrational \Rightarrow Aperiodic\ \ Signal$

   c. $x(n) = e^{j30\pi n}$

      $Angular\ \ Frequency\ \omega_o = 30\pi$

      $\omega_o = 2\pi f$

      $f = \frac{\omega_o}{2\pi} = \frac{30\pi}{2\pi} = 15$

      $Fundamental\ \ Frequency\, is\ \ rational \Rightarrow Periodic\ \ Signal$

   d. $x(n) = \sin 30n$

      $Angular\ \ Frequency\ \omega_o = 30$

      $\omega_o = 2\pi f$

      $f = \frac{\omega_o}{2\pi} = \frac{30}{2\pi} = \frac{15}{\pi}$

      $Fundamental\ \ Frequency\, is\ \ irrational \Rightarrow Aperiodic\ \ Signal$

3. Even and Odd Signals

Even Signals	Odd Signals
➢ Symmetric Signals ➢ Satisfies $x(t) = x(-t)$; $x(n) = x(-n)$ $x_e(t) = \dfrac{x(t)+x(-t)}{2}$ $x_e(n) = \dfrac{x(n)+x(-n)}{2}$	➢ Asymmetric Signals ➢ Satisfies $x(t) \neq x(-t)$; $x(n) \neq x(-n)$ $x_o(t) = \dfrac{x(t)-x(-t)}{2}$ $x_0(n) = \dfrac{x(n)-x(-n)}{2}$

Expression of Even & Odd Signals.

i) <u>Continuous time signal</u>

The continuous time signal is composed of even and odd signals

$$x(t) = x_e(t) + x_o(t) \qquad \rightarrow 1$$

put $t = -t$

$$x(-t) = x_e(-t) + x_o(-t) \qquad \rightarrow 2$$

$$x(-t) = x_e(t) - x_o(t) \qquad \rightarrow 3$$

Adding the equations 1 & 3

$$x(t) + x(-t) = x_e(t) + x_o(t) + x_e(t) - x_o(t)$$

$$x(t) + x(-t) = 2x_e(t)\,;\, x_e(t) = \frac{x(t)+x(-t)}{2}$$

Subtracting the equations 1 & 3

$$x(t) - x(-t) = x_e(t) + x_o(t) - x_e(t) + x_o(t)$$

$$x(t) - x(-t) = 2x_o(t)\,;\, x_o(t) = \frac{x(t)-x(-t)}{2}$$

ii) <u>Discrete Time Signal</u>

The discrete time signal is composed of even and odd signals

$$x(n) = x_e(n) + x_o(n) \qquad \rightarrow 1$$

put $n = -n$

$$x(-n) = x_e(-n) + x_o(-n) \qquad \rightarrow 2$$

$$x(-n) = x_e(n) - x_o(n) \qquad \rightarrow 3$$

Adding the equations 1 & 3

$$x(n) + x(-n) = x_e(n) + x_o(n) + x_e(n) - x_o(n)$$

$$x(n) + x(-n) = 2x_e(n)\,;\, x_e(n) = \frac{x(n)+x(-n)}{2}$$

Subtracting the equations 1 & 3

$$x(n) - x(-n) = x_e(n) + x_o(n) - x_e(n) + x_o(n)$$

$$x(n) - x(-n) = 2x_o(n)\,;\, x_o(n) = \frac{x(n)-x(-n)}{2}$$

Determine the even and odd components

1. $x(t)=\cos t+\sin t+\cos t\sin t$

$$x(t)=\cos t+\sin t+\cos t\sin t$$
$$put\ t=-t$$
$$x(-t)=\cos(-t)+\sin(-t)+\cos(-t)\sin(-t)$$
$$x(-t)=\cos t-\sin t-\cos t\sin t$$
$$x_e(t)=\frac{x(t)+x(-t)}{2}=x_e(t)=\frac{\cos t+\sin t+\cos t\sin t+\cos t-\sin t-\cos t\sin t}{2}$$
$$x_e(t)=\frac{2\cos t}{2}=\cos t$$
$$x_o(t)=\frac{x(t)-x(-t)}{2}=\ x_o(t)=\frac{\cos t+\sin t+\cos t\sin t-\cos t+\sin+\cos t\sin t}{2}$$
$$x_o(t)=\frac{2\sin t+2\cos t\sin t}{2}=\sin t+\sin t\cos t=x_o(t)=\sin t+\frac{\sin 2t}{2}$$

2. $$x(n)=\{-2,1,\underset{\uparrow}{2},-1,3\}$$

n	x(n)	x(-n)	$x_e(n)=\frac{x(n)+x(-n)}{2}$	$x_o(n)=\frac{x(n)-x(-n)}{2}$
-2	-2	3	0.5	-2.5
-1	1	-1	0	1
0	2	2	2	0
1	-1	1	0	-1
2	3	-2	0.5	2.5

$$x_e(n)=\{0.5,0,\underset{\uparrow}{2},0,0.5\}$$
$$x_o(n)=\{-2.5,1,\underset{\uparrow}{0},-1,2.5\}$$

4. Energy and Power Signals

Energy Signals	Power Signals
➢ Energy= Finite ➢ Power=Zero $E = \lim_{T\to\infty} \int_{-T}^{T} \lvert x(t)\rvert^2 dt$ $E = \lim_{N\to\infty} \sum_{n=-N}^{N} \lvert x(n)\rvert^2$	➢ Energy= Infinite ➢ Power= Finite $P = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \lvert x(t)\rvert^2 dt$ $P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \lvert x(n)\rvert^2$

Neither energy nor power signals

The signals that do not satisfy the conditions of either energy or power signals are called neither energy nor power signals.

The energy of a signal x(t) is defined as, $E = \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2 dt$

The power of a signal x(t) is defined as,

$$\therefore \int_{-T}^{T} |X(t)|^2 \, dt = \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} 1^2 dt = [t]_{-\frac{T_0}{2}}^{\frac{T_0}{2}} = \frac{T_0}{2} - \left(-\frac{T_0}{2}\right) = T_0 \qquad P = \underset{T\to\infty}{Lt} \frac{1}{2T} \times E$$

In equation when E=constant,

$$P = E \times \underset{T\to\infty}{Lt} \frac{1}{2T} = E \times \frac{1}{2\times\infty} = E \times 0 = 0$$

From the above analysis , we can say that when a signal has finite energy the power will be zero. Also , from the above analysis we can say that the power is finite only when energy is infinite.

Determine the power and energy for the following continuous time signals.

a) x(t)=e^-2t^ u(t)

$$\text{Energy : } E = \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2 dt \quad \text{Power : } P = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt$$

$$\therefore \int_{-T}^{T} |X(\text{t})|^2 dt = \int_{0}^{T} (|e^{-2t}|)^2 dt = \int_{0}^{T} (e^{-2t})^2 dt = \int_{0}^{T} e^{-4t} dt = \left[\frac{e^{-4t}}{-4}\right]_0^T = \left[\frac{e^{-4t}}{-4} - \frac{e^0}{-4}\right]$$

$$\therefore \int_{-T}^{T} |X(\text{t})|^2 dt = \left[\frac{1}{4} - \frac{e^{-4T}}{4}\right]$$

$$Energy, E = \underset{T\to\infty}{Lt} \int_{-T}^{T} |\text{x(t)}|^2 dt = \underset{T\to\infty}{Lt} \left[\frac{1}{4} - \frac{e^{-4T}}{4}\right] = \frac{1}{4} - \frac{e^{-\infty}}{4} = \frac{1}{4} - \frac{0}{4} = \frac{1}{4} \; joules$$

$$\text{Power, P} = \underset{T\to\infty}{Lt} \int_{-T}^{T} |\text{x(t)}|^2 dt = \underset{T\to\infty}{Lt} \frac{1}{2T} \left[\frac{1}{4} - \frac{e^{-4T}}{4}\right] = \frac{1}{\infty}\left[\frac{1}{4} - \frac{e^{-\infty}}{4}\right] = 0 \times \left[\frac{1}{4} - 0\right] = 0$$

Since energy is constant and power is zero, the given signal is an **energy signal**.

b) $X(t) = 3\cos 5\Omega_0 t$

$$\text{Energy : } E = \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2 dt$$

$$\text{Power : } P = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt$$

$$\int_{-T}^{T} |X(t)|^2 \, dt = \int_{-T}^{T} (|3\cos 5\Omega_0 t|)^2 dt = \int_{-T}^{T} |(3\cos 5\Omega_0 t)^2| \, dt = \int_{-T}^{T} (3\cos 5\Omega_0 t)^2 dt$$

$$= \int_{-T}^{T} 9\cos^2 5\Omega_0 t dt = 9\int_{-T}^{T} \left(\frac{1+\cos 10\Omega_0 t}{2}\right) dt = \frac{9}{2}\int_{-T}^{T} \left(1+\cos 10\Omega_0 t\right) dt = \frac{9}{2}\left[t + \frac{\sin 10\Omega_0 t}{10\Omega_0 t}\right]_{-T}^{T}$$

$$= \frac{9}{2}\left[T + \frac{\sin 10\Omega_0 T}{10\Omega_0 t} - \left(-T + \frac{\sin 10\Omega_0(-T)}{10\Omega_0 t}\right)\right] = \frac{9}{2}\left[2T + 2\frac{\sin 10\Omega_0 T}{10\Omega_0 t}\right] = \frac{9}{2}\left[2T + 2\frac{\sin 10\frac{2\pi}{T}T}{10\frac{2\pi}{T}t}\right]$$

$$= \frac{9}{2}\left[2T + \frac{T}{10\pi}\sin 20\pi\right] = \frac{9}{2}\left[2T + \frac{T}{10\pi}\times 0\right] = 9T$$

$$Energy, E = \underset{T\to\infty}{Lt}\int_{-T}^{T}|\mathrm{X(t)}|^2 dt = \underset{T\to\infty}{Lt}\ 9T = \infty$$

$$\text{Power, P} = \underset{T\to\infty}{Lt}\frac{1}{2T}\int_{-T}^{T}|\mathrm{X(t)}|^2 dt = \underset{T\to\infty}{Lt}\frac{1}{2T}\times 9T = \underset{T\to\infty}{Lt}\frac{9}{2} = \frac{9}{2} = 4.5 watts$$

Since energy is infinite and power is constant, the given signal is a power signal.

c) $X(t) = \cos^2(\Omega_0 t)$

$$\text{Energy : } E = \lim_{T\to\infty}\int_{-T}^{T}|x(t)|^2 dt \qquad \text{Power : } P = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|x(t)|^2 dt$$

$$\therefore \int_{-T}^{T}|X(t)|^2\ dt = \int_{-T}^{T}\left(|\cos^2(\Omega_0 t)|^2\right)dt = \int_{-T}^{T}\left(\cos^2(\Omega_0 t)^2\right)dt$$

$$= \int_{-T}^{T}\left(\frac{1+\cos 2\Omega_0 t}{2}\right)^2 dt = \int_{-T}^{T}\frac{1}{4}(1+\cos 2\Omega_0 t)^2 = \frac{1}{4}\int_{-T}^{T}(1+2\cos 2\Omega_0 t+\cos^2 2\Omega_0 t)\,\mathrm{dt}$$

$$= \frac{1}{4}\int_{-T}^{T}\left(1+2\cos 2\Omega_0 t+\frac{1+\cos 4\Omega_0 t}{2}\right)dt = \frac{1}{4}\int_{-T}^{T}\frac{2+4\ \cos 2\Omega_0 t+1+\cos 4\Omega_0 t}{2}dt$$

$$= \frac{1}{8}\int_{-T}^{T}\left(3+4\cos 2\Omega_0 t+\cos 4\Omega_0 t\right)dt = \frac{1}{8}\left[3t+\frac{4\sin 2\Omega_0 t}{2\Omega_0}+\frac{\sin 4\Omega_0 t}{4\Omega_0}\right]_{-T}^{+T}$$

$$= \frac{1}{8}\left[3T-(-3T)+\frac{4\sin 2\Omega_0 T}{2\Omega_0}-\frac{4\sin 2\Omega_0(-T)}{2\Omega_0}+\frac{\sin 4\Omega_0 T}{4\Omega_0}-\frac{\sin 4\Omega_0(-T)}{4\Omega_0}\right]$$

$$= \frac{1}{8}\left[6T+\frac{8\sin 2\Omega_0 T}{2\Omega_0}+\frac{2\sin 4\Omega_0 T}{4\Omega_0}\right] = \frac{6}{8}T+\frac{\sin 2\Omega_0 T}{2\Omega_0}+\frac{\sin 4\Omega_0 T}{16\Omega_0}$$

$$= \frac{3}{4}T+\frac{\sin 2\times\frac{2\pi}{T}\times T}{2\times\frac{2\pi}{T}}+\frac{\sin 4\times\frac{2\pi}{T}\times T}{16\times\frac{2\pi}{T}} = \frac{3}{4}T+\frac{\sin 4\pi}{\frac{4\pi}{T}}+\frac{\sin 8\pi}{\frac{32\pi}{T}} = \frac{3}{4}T$$

$$Energy, E = \underset{T\to\infty}{Lt}\int_{-T}^{+T}|\mathrm{X(t)}|^2 dt = \underset{T\to\infty}{Lt}\frac{3}{4}T = \infty$$

$$power, P = \underset{T\to\infty}{Lt}\frac{1}{2T}\int_{-T}^{+T}|\mathrm{X(t)}|^2 dt = \underset{T\to\infty}{Lt}\frac{1}{2T}\times\frac{3}{4}T = \underset{T\to\infty}{Lt}\frac{3}{8} = \frac{3}{8}watt$$

d) $X(n) = \left(\frac{1}{4}\right)^n u(n)$ Here , $X(n) = \left(\frac{1}{4}\right)^n u(n)$ for all n.

$$\therefore X(n) = \left(\frac{1}{4}\right)^n = 0.25^n ; n \geq 0$$

$$Energy, E = \sum_{n=-\infty}^{\infty} |X(n)|^2 = \sum_{n=0}^{\infty} |(0.25)^n|^2 = \sum_{n=0}^{\infty} (0.25^2)^n$$

$$= \sum_{n=0}^{\infty} (0.0625)^n = \frac{1}{1\text{-}0.0625} = 1.067\ joules$$

$$power, p = \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=-N}^{N} |X(n)|^2 = \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=0}^{N} |(0.25)^n|^2$$

$$= \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=0}^{N} |(0.25)^2|^n = \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=0}^{N} (0.0625)^n$$

$$= \underset{N\to\infty}{Lt} \frac{1}{2N+1} \frac{(0.0625)^{N+1} - 1}{0.0625\text{-}1} = \frac{1}{\infty} \times \frac{0.0625^{\infty} - 1}{0.0625\text{-}1} = 0$$

Here E is finite and P is zero and so X(n) is an energy signal.

e) $X(n) = \sin\left(\frac{\pi}{3} n\right)$

$$Energy, E = \sum_{n=-\infty}^{\infty} |\mathrm{X(n)}|^2 = \sum_{n=-\infty}^{\infty} \sin^2\left(\frac{\pi}{3} n\right) = \sum_{n=-\infty}^{\infty} \frac{1\text{-}\cos\frac{2\pi}{3} n}{2}$$

$$= \frac{1}{2}\left(\sum_{n=-\infty}^{\infty}\left(1\text{-}\cos\frac{2\pi}{3} n\right)\right) = \frac{1}{2}\left(\sum_{n=-\infty}^{\infty} 1^n - \sum_{n=-\infty}^{\infty} \cos\frac{2\pi}{3} n\right) = \frac{1}{2}(\infty - 0) = \infty$$

$$Power, P = \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=-N}^{N} |\mathrm{X(n)}|^2 = \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=-N}^{N} \sin^2 \frac{\pi n}{3}$$

$$\therefore P = \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=-N}^{N} \frac{\left(1 - \cos\frac{2\pi}{3} n\right)}{2} \quad \therefore P = \underset{N\to\infty}{Lt} \frac{1}{2N+1} \frac{1}{2}\left[\sum_{n=-N}^{N} 1^n - \sum_{n=-N}^{N} \cos\frac{2\pi}{3} n\right]$$

$$= \underset{N\to\infty}{Lt} \frac{1}{2N+1} \frac{1}{2}\left[1 + 1\underbrace{+ \ldots\ldots + 1 + 1}_{} + 1\underbrace{+ \ldots\ldots + 1 + 1}_{} - 0\right]$$

$$\therefore p = \underset{N\to\infty}{Lt} \frac{1}{2N+1} \frac{1}{2}[2N+1] = \underset{N\to\infty}{Lt} \frac{1}{2} = \frac{1}{2}\ watts$$

Since P is finite and E is infinite, x(n) is a power signal.

System:

It is a physical device which is used to analysis the signal

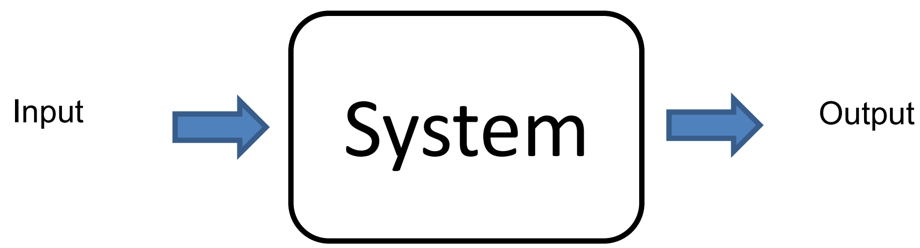
Continuous Time System

- The system which is used to analyze the **continuous time signal**

Discrete Time System

- The system which is used to analyze the **discrete time signal**

Relation between the signal and the system



$T\left[x(t)\right] = y(t)\ ;\ T\left[x(n)\right] = y(n)$

$x(t) \rightarrow Continuous\ Time\ Input\ Signal$

$x(n) \rightarrow Discrete\ Time\ Input\ Signal$

$y(t) \rightarrow Continuous\ Time\ Output\ Signal$

$y(n) \rightarrow Discrete\ Time\ Output\ Signal$

$T \rightarrow Transformation\ of\ signals$

Classification of Systems:

1. Stable & Unstable Systems
2. Static & Dynamic Systems
3. Causal & Non-Causal Systems
4. Linear & Non-Linear Systems
5. Time Variant & Time Invariant Systems

Stable & Unstable System

Condition for stability:

i) Continuous Time System $\int_{-\infty}^{\infty} |h(t)|\,dt < \infty$

ii) Discrete Time System $\sum_{n=-\infty}^{\infty} |h(n)| < \infty$

Check whether the given system is stable or unstable system

1. $h(t) = \frac{1}{Rc} e^{-t/RC} u(t)$

$$\int_{-\infty}^{\infty} |h(t)|\,dt = \int_{-\infty}^{\infty} \left| \frac{1}{Rc} e^{-t/RC} u(t) \right| dt = \frac{1}{Rc} \int_{0}^{\infty} \left| e^{-t/RC} \right| dt = \frac{1}{Rc} \left[\frac{e^{-t/RC}}{-1/RC} \right]_0^{\infty} = \frac{1}{Rc} \left[0 + RC \right] = 1$$

The given system is stable

2. $h(t) = e^{2t} u(t-1)$

$$\int_{-\infty}^{\infty} |h(t)|\,dt = \int_{-\infty}^{\infty} \left| e^{2t} u(t-1) \right| dt = \int_{1}^{\infty} \left| e^{2t} \right| dt = \left[\frac{e^{2t}}{2} \right]_0^{\infty} = \infty$$

The given system is unstable

3. $y(n) = \cos\left[x(n)\right]$

$h(n) = \cos\left[\delta(n)\right]$

$$\sum_{n=-\infty}^{\infty} |h(n)| = \sum_{n=-\infty}^{\infty} \left| \cos\left[\delta(n)\right] \right| = ...+\cos\left[\delta(-1)\right] + \cos\left[\delta(0)\right] + \cos\left[\delta(1)\right] + \cos\left[\delta(2)\right] + ... = \infty$$

The given system is unstable

Static & Dynamic System

Static System	Dynamic System
Output of the system depends on the present input only	Output of the system depends on future input only (or) future input with present input(or) future input with past input (or) future with past & present input

Causal and Non causal systems

Causal System	Non-Causal System
Output of the system depends on the past input (or) present input (or) past & present input	Output of the system depends on future input only (or) future input with past input(or) future input with present input (or) future with past & present input

The causality refers to a system that is realizable in real time. It can be shown that an LTI system is causal if and only if the impulse response is zero for n<0,

Check whether the given system is causal or non-causal systems and static or dynamic systems

1. $y(n) = x(n) - x(n-2)$

   $n = 0: \; y(0) = x(0) - x(-2)$

   $n = 1: y(1) = x(1) - x(-1)$

   $n = 2 : y(2) = x(2) - x(0)$

   The output of the system depends upon the present input and the past input; So, the system is Causal Systems, Dynamic Systems

2. $y(n) = \sum_{K=-\infty}^{n} x(K)$

   $n = -1 : y(0) = \sum_{K=-\infty}^{-1} x(K) \quad = x(-\infty) + ... + x(-1)$

   $n = 0 : y(0) = \sum_{K=-\infty}^{0} x(K) \quad = x(-\infty) + ... + x(0)$

   $n = 1 : y(1) = \sum_{K=-\infty}^{1} x(K) \quad = x(-\infty) + ... + x(0) + x(1)$

   The output of the system depends upon the present input and the past input; So, the system is Causal Systems, Dynamic System.

3. $y(n) = x(3n)$

   $n = -1 : y(-1) = x(-3)$

   $n = 0 : y(0) = x(0)$

   $n = 1 : y(1) = x(3) \;\; n = 2 : y(2) = x(6)$

   The output of the system depends upon the past, present and future input; So, the system is Non-Causal Systems, Dynamic System.

4. $y(n) = x(-n)$

$n = -2 : y(-2) = x(2)$
$n = -1 : y(-1) = x(1)$
$n = 0 : y(0) = x(0)$
$n = 1 : y(1) = x(-1)$
The output of the system depends upon the past, present and future input; So, the system is Non-Causal Systems, Dynamic System.

Linear and Non Linear systems

Linear System	Non-Linear System
Satisfies the superposition Principle	Not satisfies the superposition Principle

Superposition Principle:

The weighted sum of input is equal to the weighted sum of output

$$T[ax_1(t) + bx_2(t)] = ay_1(t) + by_2(t)$$
$$T[ax_1(n) + bx_2(n)] = ay_1(n) + by_2(n)$$

Check whether the given system is linear or non-linear system.

1. $y(n) = nx(n)$

Superposition Principle is $T[ax_1(n) + bx_2(n)] = ay_1(n) + by_2(n)$

$$ay_1(n) = anx_1(n) \; ; \; by_2(n) = bnx_2(n)$$
$$ay_1(n) + by_2(n) = anx_1(n) + bnx_2(n)$$
$$= n[ax_1(n) + bx_2(n)]$$
$$T[x(n)] = y(n) = nx(n)$$
$$T[ax_1(n) + bx_2(n)] = n[ax_1(n) + bx_2(n)]$$

The given system satisfies the superposition principle, so the system is Linear system

2. $y(n) = \log x(n)$

Superposition Principle is $T[ax_1(n) + bx_2(n)] = ay_1(n) + by_2(n)$

$$ay_1(n) = a\log x_1(n) \; ; \; by_2(n) = b\log x_2(n)$$
$$ay_1(n) + by_2(n) = a\log x_1(n) + b\log x_2(n)$$
$$T[x(n)] = y(n) = \log x(n)$$
$$T[ax_1(n) + bx_2(n)] = \log[ax_1(n) + bx_2(n)]$$
$$= \log\left[ax_1(n)\right]\log\left[bx_2(n)\right]$$

The given system not satisfies the superposition principle, so the system is Non- Linear system

3. $y(n) = x(n) - 2x(n-1)$

Superposition Principle is $T[ax_1(n)+bx_2(n)] = ay_1(n)+by_2(n)$

$$ay_1(n) = ax_1(n) - 2ax_1(n-1); by_2(n) = bx_2(n) - 2bx_2(n-1)$$

$$ay_1(n)+by_2(n) = ax_1(n) - 2ax_1(n-1) + bx_2(n) - 2bx_2(n-1)$$

$$= ax_1(n)+bx_2(n) - 2\left[ax_1(n-1)+bx_2(n-1)\right]$$

$$T[x(n)] = y(n) = x(n) - 2x(n-1)$$

$$T[ax_1(n)+bx_2(n)] = ax_1(n)+bx_2(n) - 2\left[ax_1(n-1)+bx_2(n-1)\right]$$

The given system satisfies the superposition principle, so the system is Linear system

4. $y(t) = \cos x(t)$

Superposition Principle is $T[ax_1(t)+bx_2(t)] = ay_1(t)+by_2(t)$

$$ay_1(t) = a\cos x_1(t) \; ; \; by_2(t) = b\cos x_2(t)$$

$$ay_1(t)+by_2(t) = a\cos x_1(t) + b\cos x_2(t)$$

$$T[x(t)] = y(t) = \cos x(t)$$

$$T[ax_1(t)+bx_2(t)] = \cos[ax_1(t)+bx_2(t)]$$

The given system not satisfies the superposition principle, so the system is Non- Linear system

Time Variant and Time In-variant systems

Time Variant System	Time Invariant System
➢ $y(t-T) = T[x(t-T)]$ ➢ $y(n-N) = T[x(n-N)]$	➢ $y(t-T) \neq T[x(t-T)]$ ➢ $y(n-N) \neq T[x(n-N)]$

Check whether the given system is time variant or time invariant system

1. $y(t) = \cos x(t)$

$$y(t) = \cos x(t)$$

$$Put \; t = t - T$$

$$y(t-T) = \cos x(t-T)$$

$$T[x(t)] = y(t) = \cos x(t)$$

$$Append \; (-T) \; inside \; the \; bracket$$

$$T[x(t-T)] = y(t-T) = \cos x(t-T)$$

$$y(t-T) = T[x(t-T)]$$

The given system is Time Invariant System.

2. $y(n) = ne^{x(n)}$

$$y(n) = ne^{x(n)}$$

$$Put\ n = n - N$$

$$y(n-N) = (n-N)e^{x(n-N)}$$

$$T[x(n)] = y(n) = ne^{x(n)}$$

$$Append\ (-N)\ inside\ the\ bracket$$

$$T[x(n-N)] = y(n-N) = ne^{x(n-N)}$$

$$y(n-N) \neq T[x(n-N)]$$

The given system is Time Variant System.

UNIT-2

Analysis of Continuous Time Signals

Syllabus:

Fourier Transform for Periodic Signals, Fourier Transform, -Properties, Laplace Transforms & Properties.

Session	Session Learning	Topic
1	1	Laplace Transform & types
	2	Region of convergence
2	1	Laplace Transform of Basic elementary signals
	2	Laplace Transform of transformed signals
3	1	Laplace Transform of transformed signals
	2	Properties of Laplace Transform
4	1	Properties of Laplace Transform
	2	Properties of Laplace Transform
5	1	Laplace Transform of signals using properties
	2	Laplace Transform of signals using properties
6	1	Fourier Transform & Its Properties
	2	Properties of Fourier Transform
7	1	Properties and Fourier Transform of Signals
	2	Fourier Transform of signals
8	1	Fourier Transform of signals
	2	Magnitude & Phase Spectrum
9	1	Inverse Laplace Transform
	2	Inverse Laplace Transform
10	1	Fourier Series & Its Types
	2	Exponential Fourier Series
11	1	Exponential Fourier Series
	2	Trigonometric Fourier Series
12	1	Trigonometric Fourier Series
	2	Trigonometric Fourier Series

Laplace Transform:

It is mathematical tool which is used to analysis the continuous time signal and the system

$$X(s) = L\{x(t)\} = \int_{-\infty}^{+\infty} x(t)e^{-st}dt \quad \text{Bi-Lateral Laplace Transform (Two Sided)}$$

$$X(s) = L\{x(t)\} = \int_{0}^{+\infty} x(t)e^{-st}dt \quad \text{Unilateral Laplace Transform (One Sided)}$$

$$X(s) = L\{x(t)\} = \int_{-\infty}^{0} x(t)e^{-st}dt \quad \text{Unilateral Laplace Transform (One Sided)}$$

Region of Convergence:

It is condition in which the Laplace Transform exist for the signal.
The range of σ value for which Laplace Transform X(s) converges.

$$X(s) = L\{x(t)\} = \int_{-\infty}^{+\infty} x(t)e^{-st}dt \qquad Put\ s = \sigma + j\Omega$$

$$X(s) = \int_{-\infty}^{+\infty} x(t)e^{-(\sigma + j\Omega)t}dt$$

$$X(s) = \int_{-\infty}^{+\infty} x(t)e^{-\sigma t}e^{-j\Omega t}dt \qquad If\ \ j\Omega = 0$$

$$X(s) = \int_{-\infty}^{+\infty} x(t)e^{-\sigma t}dt$$

$$X(s) = \int_{-\infty}^{+\infty} | x(t)e^{-\sigma t} |dt < \infty \qquad Condition\ \ for\ the\ exis\tan ce\ \ of\ \ Laplace\ Transform$$

Properties and Theorems of Laplace Transform

1. Amplitude Scaling

In amplitude scaling, if the amplitude (or magnitude) of a time domain signal is multiplied by a constant A, then its Laplace transform is also multiplied by the same constant.

i.e. if $\mathrm{L}\{x(t)\} = X(s)$,then

$$\mathrm{L}\{Ax(t)\} = AX(s)$$

Proof:

By definition of Laplace transform, $X(s) = L\{x(t)\} = \int_{-\infty}^{+\infty} x(t)e^{-st}dt$

$$L\{Ax(t)\} = \int_{-\infty}^{+\infty} Ax(t)e^{-st}dt \quad = A\int_{-\infty}^{+\infty} x(t)e^{-st}dt = AX(s)$$

2. Linearity

The linearity property states that, Laplace transform of weighted sum of the two or more signals is equal to similar weighted sum of Laplace transform of the individual signals.

If $L\{x_1(t)\} = X_1(s) and L\{x_2(t)\} = X_2(s), then \quad L\{a_1x_1(t) + a_2x_2(t)\} = a_1X_1(s) + a_2X_2(s)$

Proof:

By definition of Laplace transform,

$$X_1(s) = L\{x_1(t)\} = \int_{-\infty}^{+\infty} x_1(t)e^{-st}dt \qquad X_2(s) = L\{x_2(t)\} = \int_{-\infty}^{+\infty} x_2(t)e^{-st}dt$$

$$L\{a_1x_1(t) + a_2x_2(t)\} = \int_{-\infty}^{+\infty} [a_1x_1(t) + a_2x_2(t) \quad]e^{-st}dt$$

$$= a_1\int_{-\infty}^{+\infty} x_1(t)e^{-st}dt + a_2\int_{-\infty}^{+\infty} x_2(t)e^{-st}dt$$

$$= a_1X_1(s) + a_2X_2(s)$$

3. Time Differentiation

The time differentiation property states that if a causal signal X(t) is piecewise continuous, and Laplace transform of x(t) is X(s) then, Laplace transform of $\frac{d}{dt}x(t)$ is given by sX(s)-x(0). If $L\{x(t)\} = X(s), then \quad L\left\{\frac{d}{dt}x(t) = sX(s) - x(0)\right\}$; Where (0) is value of x (t) at t=0.

Proof:

By definition of Laplace transform, the Laplace transform of a causal signal is given

$$X(s) = L\{x(t)\} = \int_{0}^{+\infty} x(t)e^{-st}dt$$

$$\therefore L\left\{\frac{d}{dt}x(t)\right\} = \int_{0}^{+\infty} \frac{dx(t)}{dt}e^{-st}dt = \int_{0}^{+\infty} e^{-st}\frac{dx(t)}{dt}dt$$

$$= \left[e^{-st}\, x(t)\right]_0^{\infty} - \int_{0}^{+\infty} x(t)(-se^{-st})dt$$

$$= \left[e^{-\infty}\, x(\infty) - e^{0}x(0) + s\int_{0}^{\infty} x(t)(e^{-st})dt\right.$$

$$= s\int_{0}^{\infty} x(t)(e^{-st})dt - x(0) = sX(s) - x(0)$$

4. Time integration

The time integration property states that, if a causal signal x(t)is continuous and Laplace transform of x(t) is X(s),then the Laplace transform of $\int x(t)$ is given by, $\frac{X(s)}{s}+\frac{\left[\int x(t)dt\ \right]_{t=0}}{s}$

i.e. If $L\{x(t)\}=X(s), then \quad L\left\{\int x(t)\right\}=\frac{X(s)}{s}+\frac{\left[\int x(t)dt\ \right]_{t=0}}{s}$

Proof:

By definition of Laplace transform, the Laplace transform of a causal signal is given

$$X(s)=L\{x(t)\}=\int_0^{\infty} x(t)e^{-st}dt$$

$$\therefore L\left\{\int x(t)dt\right\}=\int_0^{\infty}\left[\int x(t)\,dt\right]e^{-st}dt$$

$$=\left[\left[\int[x(t)dt\right]\frac{e^{-st}}{-s}\right]_0^{\infty}-\int_0^{\infty}\frac{e^{-st}}{-s}\times x(t)dt$$

$$\therefore L\left\{\int x(t)dt\right\}=\left[\int x(t)\,dt\right]\Big|_{t=\infty}\frac{e^{-\infty}}{-s}-\left[\int x(t)dt\right]\Big|_{t=o}\frac{e^{0}}{-s}+\frac{1}{s}\int_0^{\infty}x(t)e^{-st}dt$$

$$=\frac{1}{s}\left[\int x(t)dt\right]\Bigg|_{t=0}+\frac{1}{s}\int_0^{\infty}x(t)e^{-st}dt$$

$$=\frac{X(s)}{s}+\frac{\left[\int x(t)dt\right]\Big|_{t=o}}{s}$$

5. Frequency Shifting

The frequency shifting property of Laplace transform says that,

$If, L\{x(t)\}=X(s), then \qquad L\left\{e^{\pm at}x(t)\right\}=X(s\,\mathrm{m}a)$

$\left[i.e. L\left\{e^{at}x(t)\right\}=X(s-a) and £\left\{e^{-at}x(t)\right\}=x(s+a)\right]$

Proof:

By definition of Laplace transform,

$$X(s)=L\{x(t)\}=\int_{-\infty}^{+\infty}x(t)e^{-st}\ dt$$

$$\therefore L\left\{e^{\pm at}x(t)\right\}=\int_{-\infty}^{+\infty}e^{\pm at}x(t)e^{-st}\ dt$$

$$= \int_{-\infty}^{+\infty} x(t)e^{-(s \mp a t)}dt \quad = X(s \mp a)$$

6. Time Shifting

The time shifting property of Laplace transform says that,

$$If, L\{x(t)\} = X(s), then \quad L\{x(t \pm) a\} = e \pm X(s)$$

$$i.e. L\{x(t+a)\} = e^{as} X(s) and L\{x(t-a)\} = e^{-as} X(s)$$

Proof:

By definition of Laplace transform,

$$X(s) = L\{x(t)\} = \int_{-\infty}^{+\infty} x(t)e^{-st} \ dt$$

$$\therefore L\{x(t \pm a)\} = \int_{-\infty}^{+\infty} x(t \pm a)e^{-st}dt = \int_{-\infty}^{+\infty} x(\tau)e^{-s(\tau \mp a)}d\tau$$

$$= \int_{-\infty}^{+\infty} x(\tau)e^{-s\tau} \times e^{\pm as}d\tau = e^{\pm as}\int_{-\infty}^{+\infty} x(\tau)e^{-st} \ d\tau$$

$$= e^{\pm as}\int_{-\infty}^{+\infty} x(\tau)e^{-st} \ dt = e^{\pm as}x(s)$$

7. Frequency Differentiation

The frequency differentiation property of Laplace transform says that,

$$If, L\{x(t)\} = X(s), then \ L\{tx(t)\} = -\frac{d}{ds}X(s)$$

Proof:

By definition of Laplace transform,

$$X(s) = L\{tx(t)\} = \int_{-\infty}^{+\infty} x(t)e^{-st} \ dt$$

On differentiating the above equation with respect to s we get,

$$\frac{d}{ds}X(s) = \frac{d}{ds}\left(\int_{-\infty}^{+\infty} x(t)e^{-st} \ dt\right)$$

$$= \int_{-\infty}^{+\infty} x(t)\left(\frac{d}{ds}e^{-st}\right)dt = \int_{-\infty}^{+\infty} x(t)(-te^{-st}) \ dt$$

$$= \int_{-\infty}^{+\infty} (-tx(t))e^{-st}dt = L\{-tx(t)\} = -L\{tx(t)\}$$

$$= L\{tx(t)\} = -\frac{d}{ds}X(s)$$

$$L\left\{\frac{1}{t}x(t)\right\} = \int_{s}^{\infty} X(s)ds$$

8. Frequency Integration

The frequency integration property of Laplace transform says that,

$$If\ \ L\{x(t)\} = X(s), then \quad L\left\{\frac{1}{t}x(t)\right\} = \int_s^{\infty} X(s)ds$$

Proof:

By definition of Laplace transform,

$$X(s) = L\{x(t)\} = \int_{-\infty}^{+\infty} x(t)e^{-st}\ dt$$

On integrating the above equation with respect to s we get,

$$\int_{-\infty}^{+\infty} x(s)\ ds = \int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty} x(t)e^{-st}\ dt\right]ds$$

$$= \int_{+\infty}^{-\infty} x(t)\left[\int_{-\infty}^{+\infty} e^{-st}\ ds\right]dt = \int_{-\infty}^{+\infty} x(t)\left[\frac{e^{-st}}{-t}\right]_s^{\infty} dt = \int_{-\infty}^{+\infty} x(t)\left[\frac{e^{-\infty}}{-t} - \frac{e^{-st}}{-t}\right]dt$$

$$= \int_{-\infty}^{+\infty} x(t)\left[0 + \frac{e^{-st}}{-t}\right]dt = \int_{-\infty}^{+\infty}\left[\frac{1}{t}x(t)\right]e^{-st}dt = L\left\{\frac{1}{t}x(t)\right\}$$

$$\therefore L\left\{\frac{1}{t}x(t)\right\} = \int_s^{\infty} X(s)ds$$

9. Time Scaling

The time scaling property of Laplace transform says that,

$$If, L\{x(t)\} = X(s), then \qquad L\{x(at)\} = \frac{1}{|a|}X\left(\frac{s}{a}\right)$$

Proof:

By definition of Laplace transform,

$$X(s) = L\{x(t)\} = \int_{-\infty}^{+\infty} x(t)e^{-st}\ dt$$

$$\therefore L\{x(at)\} = \int_{-\infty}^{\infty} x(at)e^{-st}dt = \int_{-\infty}^{+\infty} x(\tau)e^{-s(\frac{\tau}{a})}\frac{d\tau}{a}$$

$$= \frac{1}{a}\int_{-\infty}^{+\infty} x(\tau)e^{-\left(\frac{s}{a}\right)\tau} d\tau = \frac{1}{a}\times\left(\frac{s}{a}\right)$$

The above transform is applicable for positive values of “a” happens to be negative it can be proved that, $L\{x(at)\} = -\frac{1}{a}X\left(\frac{s}{a}\right)$

Hence in general, $L\{x(at)\} = \frac{1}{|a|} X\left(\frac{s}{a}\right)$ for both positive and negative values of "a"

10. Periodicity

The periodicity property of Laplace transform says that,

If $x\{(t)\} = x(t+nT), \textit{and } x_1(t) \textit{ be one period of } x(t),$

$$L\{x(t+nT)\} = \frac{1}{1-e^{-st}} \int_0^T x_1(t)e^{-st}dt$$

Proof:

By definition of Laplace transform, $L\{x(t+nT)\} = \int_0^\infty x(t+nT)e^{-st}dt$

$$= \int_0^T x_1(t)e^{-st}dt + \int_T^{2T} x_1(t-T)e^{-s(t+T)}dt + \int_{2T}^{3T} x_1(t-2T)e^{-s(t+2T)}dt + \text{K} \ldots\ldots\ldots\ldots + \int_{pT}^{(p+1)T} x_1(t-pT)e^{-s(t+pT)}dt + \ldots\ldots\ldots$$

$$\therefore L\{x(t+nT)\} = \sum_{p=0}^{\infty} \int_{pT}^{(p+1)T} x_1(t-pT)e^{-s(t+pT)}dt$$

$$= \sum_{p=0}^{\infty} \int_0^T x_1(t)e^{-st}e^{-psT}dt = \int_0^T x_1(t)e^{-st}\left(\sum_{p=0}^{\infty} e^{-psT}\right)dt$$

$$= \int_0^T x_1(t)e^{-st}\left(\sum_{p=0}^{\infty} e^{-sT}\right)^p dt = \int_0^T x_1(t)e^{-st}\left(\frac{1}{1-e^{-sT}}\right)dt$$

$$= \left(\frac{1}{1-e^{-sT}}\right)\int_0^T x_1(t)e^{-st}dt$$

11. Initial value Theorem

The initial value theorem states that, if x(t) and its derivative are Laplace transformable then, $\underset{t\to 0}{Lt}\, x(t) = \underset{s\to\infty}{Lt}\, sX(s)$

Proof:

We know that, $L\left\{\frac{dx(t)}{dt}\right\} = sX(s) - x(0)$

On taking limit $s \to \infty$ on both sides of the above equation we get,

$$\underset{s\to\infty}{Lt}\, L\left\{\frac{dx(t)}{dt}\right\} = \underset{s\to\infty}{Lt}\left[sX(s) - x(0)\right]$$

$$\underset{s\to\infty}{Lt}\int_0^\infty \frac{dx(t)}{dt}e^{-st}dt = \underset{s\to\infty}{Lt}\left[sX(s) - x(0)\right]$$

$$\int_0^\infty \frac{dx(t)}{dt}\left(\underset{s\to\alpha}{Lt}\, e^{-st}\right)dt = \left(\underset{s\to\alpha}{Lt}\, sX(s)\right) - x(0)$$

$$0 = \underset{s \to \infty}{Lt}\ sX(s) - x(0) \quad \therefore x = (0)\ \underset{s \to \infty}{Lt}\ sX(s)$$

$$\therefore \underset{t \to 0}{Lt}\, x(t) = \underset{s \to \infty}{Lt}\ sX(s)$$

12. Final value Theorem:

The final value theorem states that, if x (t) and its derivative is Laplace transformable then, $\underset{t \to \infty}{Lt}\ x(t) = \underset{s \to 0}{Lt}\ sX(s)$

Proof:

We know that, $L\left\{\dfrac{dx(t)}{dt}\right\} = sX(s) - x(0)$

On taking limit $s \to 0$ on both sides of the above equation we get,

$$\underset{s \to 0}{Lt}\ L\left\{\frac{dx(t)}{dt}\right\} = \underset{s \to 0}{Lt}\left[sX(s) - x(0)\right]$$

$$\underset{s \to 0}{Lt}\int_0^\infty \frac{dx(t)}{dt} e^{-st} dt = \underset{s \to 0}{Lt}\left[sX(s) - x(0)\right]$$

$$\int_0^\infty \frac{dx(t)}{dt}\left(\underset{s \to 0}{Lt}\ e^{-st}\right) dt = \left(\underset{s \to 0}{Lt}\ sX(s)\right) - x(0)$$

$$\int_0^\infty \frac{dx(t)}{dt} dt = \underset{s \to 0}{Lt}\ e^{-st}\ \underset{s \to 0}{Lt}\ sX(s) - x(0)$$

$$\left[x(t)\right]_0^\infty = \underset{s \to 0}{Lt}\ sX(s) - x(0)$$

$$x(\infty) - x(0) = \underset{s \to 0}{Lt}\ sX(s) - x(0)$$

$$\therefore x(\infty) = \underset{s \to 0}{Lt}\ sX(s)$$

$$\underset{t \to \infty}{Lt}\ x(t) = \underset{s \to 0}{Lt}\ sX(s)$$

13. Convolution Theorem:

The convolution theorem of Laplace transform says that, Laplace transform of convolution of two signals is given by the product of the Laplace transform of the individual signals. If $L\{x_1(t)\} = X_1(s)\ and\ L\{x_2(t)\} = X_2(s)\ then,$

$$L\{x_1(t) * x_2(t)\} = X_1(s)\ X_2(s) \quad x_1(t) * x_2(t) = \int_{-\infty}^{+\infty} x_1(\lambda) x_2(t-\lambda) d\lambda$$

Where, λ is Dummy variable used for integration.

Proof:

Let $x_1 and x_2\ be\ two\ time\ domain\ signals.$

By definition of Laplace transform,

$$X_1(s) = L\{x_1(t)\} = \int_{-\infty}^{+\infty} x_1(t) e^{-st} \qquad X_2(s) = L\{x_2(t)\} = \int_{-\infty}^{+\infty} x_2(t) e^{-st}\ dt$$

Now by definition of Laplace transform,

$$L\{x_1(t) * x_2(t)\} = \int_{-\infty}^{+\infty} [x_1(t) * x_2(t)] e^{-st} dt$$

$$= \int_{t=-\infty}^{t=+\infty} \left[\int_{\lambda=-\infty}^{\lambda=+\infty} [x_1(\lambda) x_2(t-\lambda) d\lambda] \right] e^{-st} dt$$

$$= \int_{t=-\infty}^{t=+\infty} \int_{t=-\infty}^{t=+\infty} x_1(\lambda) x_2(t-\lambda) d\lambda\ e^{s\lambda} e^{-s\lambda} e^{-s\lambda} e^{-st} dt$$

$$= \int_{t=-\infty}^{t=+\infty} \int_{\lambda=-\infty}^{\lambda=+\infty} x_1(\lambda) x_2(t-\lambda) e^{-s\lambda}\ e^{-s(t-\lambda)} d\lambda dt$$

$$= \int_{M=-\infty}^{M=+\infty} \int_{\lambda=-\infty}^{\lambda=+\infty} x_1(\lambda) x_2(M) e^{-s\lambda}\ e^{-sM} d\lambda\ dM$$

$$= \int_{-\infty}^{+\infty} x_1(t) e^{-st} dt \times \int_{-\infty}^{+\infty} x_2(t) e^{-st} dt$$

$$= X_1(s) X_2(s)$$

$$\therefore L\{x_1(t) * x_2(t)\} = X_1(s)\ X_2(s)$$

Table: Properties of Laplace Transform

Property	Time Domain Signal	s-domain signal
Amplitude scaling	$AX(t)$	$AX(s)$
Linearity	$a_1x_1(t) \pm a_2x_2(t)$	$a_1X_1(s) \pm a_2X_2(s)$
Time differentiation	$\frac{d}{dt}x(t)$	$sXs(s) - x(0)$
	$\frac{d^m}{dt^m}x(t)$ Where m=1, 2, 3……	$s^m X(s) \sum_{k=1}^{m} s^{m-k} \left. \frac{d^{(k-1)}x(t)}{dt^{k-1}} \right\|_{t=0}$
Time integration	$\int x(t)dt$	$\frac{X(s)}{s} + \frac{\left[\int x(t)\,dt\ \right]_{t=0}}{s}$
	$\int .....\int x(t)(dt)^m$ Where m=1, 2, 3……	$\frac{X(s)}{s^m} + \sum_{k=1}^{m} \frac{1}{s^{m-k+1}} \left\| \int .....\int x(t)(dt)k \right\|_{t=0}$
Frequency shifting	$e^{\pm at}X(t)$	$X(s \mp a)$
Time shifting	$x(t \pm a)$	$e^{\pm as}X(s)$
Frequency differentiation	$tx(t)$	$-\frac{dX(s)}{ds}$
	$t^m x(t)$ Where m=1, 2, 3……	$-(1)^m \frac{dm}{ds^m} X(s)$
Frequency integration	$\frac{1}{t}x(t)$	$\int_s^\infty X(s)ds$
Time scaling	$x(at)$	$\frac{1}{\|a\|} \times \left(\frac{s}{a}\right)$
Periodicity	$x(t+mT)$ Where m=1, 2, 3…… T=Period	$\frac{1}{1-e^{-st}} \int_0^T x_1(t)e^{-st}dt$ Where, $x_1(t)$ is one period of $x(t)$
Initial value theorem	$\underset{t\to 0}{Lt}\ x(t) = x(0)$	$\underset{s\to \infty}{Lt}\ sX(s)$
Final value theorem	$\underset{t\to \infty}{Lt}\ x(t) = x(\infty)$	$\underset{s\to 0}{Lt}\ sX(s)$
Convolution theorem	$\int_{-\infty}^{+\infty} x_1(\lambda)x_2(t-\lambda)d\lambda$	$X_1(s)X_2(s)$

Determine the Laplace transform for the following signals

1. $x(t) = \delta(t)$

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt = \int_{-\infty}^{\infty} \delta(t)e^{-st}dt \qquad \delta(t) = \begin{cases} 1; t = 0 \\ 0; t \neq 0 \end{cases}$$

$$= \delta(0)e^{-s(0)} = 1$$

2. $x(t) = u(t)$

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt = \int_{-\infty}^{\infty} u(t)e^{-st}dt \qquad u(t) = \begin{cases} 1; t \geq 0 \\ 0; t < 0 \end{cases}$$

$$= \int_{0}^{\infty} e^{-st}dt = \left[\frac{e^{-st}}{-s}\right]_0^{\infty}$$

$$= \left[\frac{e^{-s(\infty)}}{-s} - \frac{e^{-s(0)}}{-s}\right] = \left[0 - \frac{1}{s}\right] = \frac{1}{s}$$

3. $x(t) = r(t)$

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt = \int_{-\infty}^{\infty} r(t)e^{-st}dt = \int_{0}^{\infty} te^{-st}dt \qquad r(t) = \begin{cases} t; t \geq 0 \\ 0; t < 0 \end{cases}$$

$$= \left[-t\frac{e^{-st}}{s} - \frac{e^{-st}}{s^2}\right]_0^{\infty} \qquad u = t \; ; v = e^{-st}$$

$$= \left[(0-0) - \left(0 - \frac{1}{s^2}\right)\right] \qquad u^{'} = 1 \; ; v = \frac{e^{-st}}{-s}$$

$$= \frac{1}{s^2} \qquad u^{''} = 0; \; v = \frac{e^{-st}}{s^2}$$

4. $x(t) = \cos\omega_o t u(t)$

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt = \int_{-\infty}^{\infty} \cos\omega_o t u(t)e^{-st}dt \qquad \cos\omega_o t = \frac{e^{j\omega_o t} + e^{-j\omega_o t}}{2}$$

$$= \int_{0}^{\infty} \cos\omega_o t e^{-st}dt = \int_{0}^{\infty}\left(\frac{e^{j\omega_o t} + e^{-j\omega_o t}}{2}\right)e^{-st}dt$$

$$= \frac{1}{2}\int_{0}^{\infty}\left[e^{j\omega_o t}e^{-st} + e^{-j\omega_o t}e^{-st}\right]dt = \frac{1}{2}\int_{0}^{\infty}\left[e^{-(s-j\omega_o)t} + e^{-(s+j\omega_o)t}\right]dt$$

$$=\frac{1}{2}\left[\frac{e^{-(s-j\omega_o)t}}{-(s-j\omega_o)}+\frac{e^{-(s+j\omega_o)t}}{-(s+j\omega_o)}\right]_0^{\infty}=\frac{1}{2}\left[(0+0)-\left(\frac{1}{-(s-j\omega_o)}+\frac{1}{-(s+j\omega_o)}\right)\right]$$

$$=\frac{1}{2}\left(\frac{1}{(s-j\omega_o)}+\frac{1}{(s+j\omega_o)}\right)=\frac{1}{2}\left(\frac{s+j\omega_o+s-j\omega_o}{(s^2+\omega_o{}^2)}\right)=\frac{1}{2}\left(\frac{2s}{(s^2+\omega_o{}^2)}\right)=\frac{s}{(s^2+\omega_o{}^2)}$$

5. $x(t)=e^{at}\sin btu(t)$

$$X(s)=\int_{-\infty}^{\infty}x(t)e^{-st}dt=\int_{-\infty}^{\infty}e^{at}\sin btu(t)e^{-st}dt$$

$$using\ \ frequency\ \ shifting\ \ property\ \ L[e^{at}x(t)]=X(s-a)$$

$$L[\sin bt]=\frac{b}{(s^2+b^2)}$$

$$L[e^{at}\sin bt]=\frac{b}{((s-a)^2+b^2)}$$

S.No.	Signal	Laplace Transform
1.	$\delta(t)$	1
2.	$u(t)$	$\frac{1}{s}$
3.	$e^{-at}u(t)$	$\frac{1}{s+a}$
4.	$e^{at}u(t)$	$\frac{1}{s-a}$
5.	$r(t)$	$\frac{1}{s^2}$
6.	$e^{-at}r(t)$	$\frac{1}{(s+a)^2}$
7.	$e^{at}r(t)$	$\frac{1}{(s-a)^2}$
8.	$\cos at$	$\frac{s}{s^2+a^2}$
9.	$\sin at$	$\frac{a}{s^2+a^2}$
10.	$e^{-bt}\cos at$	$\frac{s+b}{(s+b)^2+a^2}$
11.	$e^{bt}\cos at$	$\frac{s-b}{(s-b)^2+a^2}$
12.	$e^{-bt}\sin at$	$\frac{a}{(s+b)^2+a^2}$

13.	$e^{bt}\sin at$	$\dfrac{a}{(s-b)^2+a^2}$

Inverse Laplace Transform Types

Type-1	$\dfrac{1}{(s+1)(s+2)}$	$\dfrac{A}{(s+1)}+\dfrac{B}{(s+2)}$
Type-2	$\dfrac{1}{(s+1)(s+2)^2}$	$\dfrac{A}{(s+1)}+\dfrac{B}{(s+2)}+\dfrac{C}{(s+2)^2}$
Type-3	$\dfrac{1}{(s+1)(s^2+2)}$	$\dfrac{A}{(s+1)}+\dfrac{Bs+c}{(s^2+2)}$

Determine the Inverse Laplace Transform

1. $$X(s)=\frac{1}{(s+2)(s+3)}$$

$$=\frac{A}{(s+2)}+\frac{B}{(s+3)}=\frac{A(s+3)+B(s+2)}{(s+2)(s+3)} \qquad \Rightarrow 1=A(s+3)+B(s+2)$$

$$Put\ \ s=-2 \Rightarrow 1=A(-2+3)+B(-2+2) \Rightarrow\ A=1$$

$$Put\ \ s=-3 \Rightarrow 1=A(-3+3)+B(-3+2) \Rightarrow\ B=-1$$

$$X(s)=\frac{1}{(s+2)}+\frac{-1}{(s+3)}$$

$$Taking\ \ Inverse\ \ Laplace\ Transform\ \ on\ both\ \ sides$$

$$x(t)=e^{-t}u(t)-e^{-2t}u(t)$$

2. $$X(s)=\frac{s}{(s+2)(s+3)^2}$$

$$=\frac{A}{(s+2)}+\frac{B}{(s+3)}+\frac{C}{(s+3)^2}=\frac{A(s+3)^2+B(s+2)(s+3)+C(s+2)}{(s+2)(s+3)^2}$$

$$\Rightarrow s=A(s+3)^2+B(s+2)(s+3)+C(s+2)$$

$$Put\ \ s=-2 \Rightarrow -2=A(-2+3)^2+B(-2+2)(-2+3)+C(-2+2) \Rightarrow\ A=-2$$

$$Put\ \ s=-3 \Rightarrow -3=A(-3+3)^2+B(-3+2)(-3+3)+C(-3+2) \Rightarrow\ C=3$$

$$Put\ \ s=0 \Rightarrow 0=A(0+3)^2+B(0+2)(0+3)+C(0+2) \Rightarrow\ 9A+6B+2C=0$$

$$\Rightarrow 9(-2)+6(B)+2(3)=0$$

$$\Rightarrow -18+6B+6=0$$

$$\Rightarrow 6B = 12 \Rightarrow B = 2$$

$$X(s) = \frac{-2}{(s+2)} + \frac{2}{(s+3)} + \frac{3}{(s+3)^2}$$

Taking Inverse Laplace Transform on both sides

$$x(t) = -2e^{-2t}u(t) + 2e^{-3t}u(t) + 3te^{-3t}u(t)$$

Fourier Transform

Let, $x(t) = \text{Continuous time signal}$

$X(j\Omega) = \text{Fourier Transform of } x(t)$

$$F[x(t)] = X(j\Omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\Omega t}\,dt$$

Conditions for Existence of Fourier Transform

The Fourier transform of x(t) exists if it satisfies the following Dirichlet's condition.

1. The x (t) should be absolutely integrable. i.e., $\int_{-\infty}^{+\infty} x(t)e^{-j\Omega t}\,dt < \infty$
2. The x(t) should have a finite number of maxima and minima within any finite interval.
3. The x(t) can have a finite number of discontinuities within any interval.

Definition of Fourier Transform

The inverse Fourier transform of X(jΩ) is defined as,

$$x(t) = F^{-1}\{X(j\Omega)\} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\Omega)e^{j\Omega t}\,d\Omega$$

The signals x(t) and X(jΩ) are called Fourier transform pair can be expressed as shown below:

$$x(t) \underset{F^{-1}}{\overset{F}{\rightleftarrows}} X(j\Omega)$$

Frequency Spectrum Using Fourier Transform

The $X(j\Omega)$ is a complex function of **Ω**.Hence it can be expressed as a sum of real part and imaginary part as shown as below:

$$\therefore X(j\Omega) = X_r(j\Omega) + jX_i(j\Omega)$$

Where, $X_r(j\Omega)$=Real part of $X(j\Omega)$; $X_i(j\Omega)$ =Imaginary part of $X(j\Omega)$

The magnitude of $X(j\Omega)$ is called **Magnitude spectrum,** $|X(j\Omega)| = \sqrt{X_r^2(j\Omega) + X_i^2(j\Omega)}$

The phase Spectrum of X(jΩ) is called **Phase spectrum.** $\angle X(j\Omega) = \tan^{-1}\left(\frac{X_i(j\Omega)}{X_R(j\Omega)}\right)$

The magnitude spectrum will always have symmetry band phase spectrum will have odd symmetry. The magnitude and phase spectrum together called **frequency spectrum.**

Comparison of Fourier series and Fourier transform

Fourier Series	Fourier Transform
1.Defined only for periodic signals	1. Defined for both periodic and aperiodic signals
2.The spectrum is discrete	2.The spectrum is continuous
3.Magnitude spectrum and phase spectrum are plotted by taking "magnitude/phase" of a signal versus harmonic order "n"	3.Magnitude spectrum and phase spectrum are plotted by taking "magnitude/phase" of a signal versus frequency "**Ω**"
4. Parseval's relation of Fourier series is used to calculate power spectral density of a periodic signal x(t).	4. Parseval's relation of Fourier transform is used to calculate energy spectral density of the signal x(t).

Properties of Fourier Transform

1. Linearity

The linearity property of Fourier transform says that,

$$F[ax_1(t)+bx_2(t)] = aX_1(j\Omega)+bX_2(j\Omega)$$

Proof:

By definition of Fourier transform,

$$X_1(j\Omega) = \int_{-\infty}^{+\infty} x_1(t)e^{-j\Omega t}dt \quad X_2(j\Omega) = \int_{-\infty}^{+\infty} x_2(t)e^{-j\Omega t}dt$$

$$F[ax_1(t)+bx_2(t)] = \int_{-\infty}^{+\infty} [a_1x_1(t)+a_2x_2(t)]e^{-j\Omega t}dt$$

$$= \int_{-\infty}^{+\infty} (a_1x_1(t)e^{-j\Omega t}+a_2x_2(t)e^{-j\Omega t})dt$$

$$= \int_{-\infty}^{+\infty} a_1x_1(t)e^{-j\Omega t}dt + \int_{-\infty}^{\infty} a_2x_2(t)e^{-j\Omega t}dt$$

$$aX_1(j\Omega) + bX_2(j\Omega)$$

2. Time shifting

Time shifting property of Fourier transform says that,

$$F[x(t)] = X(j\Omega) \text{ then } F[x(t-t_o)] = e^{-j\Omega t_o} X(j\Omega)$$

Proof:

$$F[x(t-t_o)] = \int_{-\infty}^{+\infty} x(t-t_0) e^{-j\Omega t} dt \qquad t - t_0 = \tau \Rightarrow t = \tau + t_o \;\; dt = d\tau$$

$$= \int_{-\infty}^{+\infty} x_2(\tau) e^{-j\Omega(\tau + t_0)} d\tau = \int_{-\infty}^{+\infty} x_2(\tau) e^{-j\Omega\tau} e^{-j\Omega t_0} d\tau$$

$$= e^{-j\Omega t_0} \int_{-\infty}^{+\infty} x_2(\tau) e^{-j\Omega\tau} d\tau = e^{-j\Omega t_o} X(j\Omega)$$

3. Time Scaling

The time scaling property of Fourier transform says that,

$$F[x(t)] = X(j\Omega) \text{ then } F[x(at)] = \frac{1}{|a|} X\left(\frac{j\Omega}{a}\right)$$

Proof:

By definition of Fourier transform, $F[x(t)] = \int_{-\infty}^{+\infty} x(t) e^{-j\Omega t} dt$

$$F[x(at)] = \int_{-\infty}^{+\infty} x(at) e^{-j\Omega t} dt \qquad at = \tau \Rightarrow t = \frac{\tau}{a} \qquad adt = d\tau \Rightarrow dt = \frac{d\tau}{a}$$

$$= \frac{1}{a} \int_{-\infty}^{+\infty} x(\tau) e^{-j\Omega\frac{\tau}{a}} d\tau \quad = \frac{1}{|a|} X\left(\frac{j\Omega}{a}\right)$$

4. Time Reversal

The time reversal property of Fourier transform says that,

$$F[x(t)] = X(j\Omega) \text{ then } F[x(-t)] = X(-j\Omega)$$

Proof:

By definition of Fourier transform, $F[x(t)] = \int_{-\infty}^{+\infty} x(t) e^{-j\Omega t} dt$

$$F[x(-t)] = \int_{-\infty}^{+\infty} x(-t)e^{-j\Omega t}dt \qquad -t = \tau \Rightarrow t = -\tau \quad -dt = d\tau$$

$$= \int_{-\infty}^{+\infty} x(\tau)e^{j\Omega\tau}d\tau$$

$$= \int_{-\infty}^{+\infty} x(\tau)e^{-(-j\Omega)\tau}d\tau = X(-j\Omega)$$

5. Conjugation

The conjugation property of Fourier transform says that,

$$F[x(t)] = X(j\Omega) \text{ then } F\{x^*(t)\} = X^*(-j\Omega)$$

Proof:

By definition of Fourier transform, $F\{x(t) = X(j\Omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\Omega t}dt$

$$F[x^*(t)] = \int_{-\infty}^{+\infty} x^*(t)e^{-j\Omega t}dt$$

$$= \left[\int_{-\infty}^{+\infty} x(t)e^{j\Omega t}dt\right]^*$$

$$= \left[\int_{-\infty}^{+\infty} x(t)e^{-j(-\Omega)t}dt\right]^*$$

$$= X^*(-j\Omega)$$

6. Frequency Shifting

The frequency shifting property of Fourier transform says that,

$$F[x(t)] = X(j\Omega) \text{ then } F\{e^{j\Omega_0 t}x(t) = X(j(\Omega - \Omega_0)$$

Proof:

By definition of Fourier transform,

$$F\{x(t)\} = X(j\Omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\Omega t}dt$$

$$F\{e^{j\Omega_0 t}x(t)\}$$

$$= \int_{-\infty}^{+\infty} e^{j\Omega_0 t}x(t)e^{-j\Omega t}dt$$

$$= \int_{-\infty}^{+\infty} x(t)e^{j\Omega_0 t}e^{-j\Omega t}dt$$

$$= \int_{-\infty}^{+\infty} x(t) e^{-j(\Omega - \Omega 0)t} dt$$

$$= X(j\Omega - \Omega_0)$$

7. Time Differentiation

The differentiation property of Fourier transform says that,

$$F[x(t)] = X(j\Omega) \quad \text{then} \quad F\left\{\frac{d}{dt} x(t)\right\} = j\Omega X(j\Omega)$$

Proof:

By definition of inverse Fourier transform,

$$x(t) \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\Omega) e^{j\Omega t} d\Omega$$

On differentiating the above equation we get,

$$\frac{dx(t)}{dt} = \frac{dt}{d} \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\Omega) e^{j\Omega t} d\Omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\Omega) \frac{d}{dt} e^{j\Omega t} d\Omega = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\Omega) j\Omega e^{j\Omega t} d\Omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} ([j\Omega) X(j\Omega)] e^{j\Omega t} d\Omega = F^{-1}\{j\Omega X j\Omega\}$$

On taking Fourier transform of the above equation we get,

$$F\left\{\frac{dx(t)}{dt}\right\} = j\Omega X(j\Omega)$$

8. Time Integration

The integration property of Fourier transform says that,

$$F[x(t)] = X(j\Omega) \text{ and X(0) = 0 then } F\left[\int_{-\infty}^{t} x(\tau) d\tau\right] = \frac{1}{j\Omega} X(j\Omega)$$

Proof:

Consider a continuous time signal x(t).Let X(j**Ω**)be Fourier transform of x(t).since integration and differentiation are inverse operations "t" can be expressed as shown below:

$$\frac{d}{dt}\left[x(\tau) d\tau\right] = x(t)$$

On taking Fourier transform of the above equation we get,

$$F\left\{\frac{d}{dt}\left[\int_{-\infty}^{t} x(\tau)d\tau\right]\right\} = F\{x(t)\}$$

$$j\Omega F\left[\int_{-\infty}^{t} x(\tau)d\tau\right] = F\{x(t)\}$$

$$F\left[\int_{-\infty}^{t} x(\tau)d\tau\right] = \frac{1}{j\Omega}X(j\Omega)$$

9. Frequency Differentiation

The frequency differentiation property of Fourier transform says that,

$$F[x(t)] = X(j\Omega) \text{ then } F\{tx(t)\} = j\frac{d}{d\Omega}X(j\Omega)$$

Proof:

By definition of Fourier transform,

$$\frac{d}{d\Omega}X(j\Omega) = \frac{d}{d\Omega}\left(\int_{-\infty}^{+\infty} x(t)e^{-j\Omega t}dt\right) = \int_{-\infty}^{+\infty} x(t)\left(\frac{d}{d\Omega}e^{-j\Omega t}\right)dt$$

$$= \int_{-\infty}^{+\infty} x(t)\left(-jte^{-j\Omega t}\right)dt = \frac{1}{j}tx(t)e^{-j\Omega t}dt$$

$$= \frac{1}{j}F\{tx(t)\}$$

$$\therefore F\{tx(t)\} = j\frac{d}{d\Omega}X(j\Omega)$$

10. Convolution theorem

Convolution theorem of Fourier transform says that, Fourier transform of convolution for two signals is given by the product of the Fourier transform of the individual signals.

$$F[x(t)] = X(j\Omega) \text{ then, } F[x_1(t) * x_2(t)] = X_1(s)X_2(s)$$

The equation is also known as convolution property of Fourier transform.

By definition of convolution of continuous time signals,

$$x_1(t) * x_2(t) = \int_{\tau=-\infty}^{\tau=-\infty} x_1(\tau)x_2(t-\tau)d\tau$$

Proof:

Using definition Fourier transform we can write,

$$F\{x_1(t) * x_2\} = \int_{t=-\infty}^{t=+\infty} [x_1(t) * x_2(t)]e^{-j\Omega t}dt$$

$$= \int_{\tau=-\infty}^{\tau=-\infty}\left[\int_{\tau=-\infty}^{\tau=-\infty} x_1(t)x_2(t-\tau)d\tau\right]e^{-j\Omega t}dt \qquad t-\tau = u \Rightarrow dt = du$$

$$= \int_{t=-\infty}^{t=-\infty} \left[\int_{\tau=-\infty}^{\tau=-\infty} x_1(\tau) x_2(u) d\tau \right] e^{-j\Omega(\tau+u)} dt$$

$$= \int_{\tau=-\infty}^{\tau=+\infty} x_1(\tau) e^{-j\Omega\tau} d\tau \int_{u=-\infty}^{u=+\infty} x_2(u) e^{-j\Omega u} d\tau$$

$$= X_1(j\Omega) X_2(j\Omega)$$

1. Determine the Fourier Transform for the given signal and draw its phase and magnitude spectrum.
   $x(t) = e^{-at}u(t)$

$$F[x(t)] = \int_{-\infty}^{+\infty} x(t) e^{-j\Omega t} dt$$

$$= \int_{-\infty}^{+\infty} e^{-at} u(t) e^{-j\Omega t} dt = \int_{0}^{+\infty} e^{-at} e^{-j\Omega t} dt$$

$$= \int_{0}^{+\infty} e^{-(a+j\Omega)t} dt = \left[\frac{e^{-(a+j\Omega)t}}{-(a+j\Omega)} \right]_0^{\infty}$$

$$= \left[\frac{e^{-(a+j\Omega)\infty}}{-(a+j\Omega)} - \frac{e^{-(a+j\Omega)0}}{-(a+j\Omega)} \right] = \frac{1}{(a+j\Omega)}$$

$$= \frac{1}{(a+j\Omega)} x \frac{(a-j\Omega)}{(a-j\Omega)} = \frac{(a-j\Omega)}{(a^2+\Omega^2)}$$

$$X_R(j\Omega) = \frac{a}{(a^2+\Omega^2)} \qquad X_I(j\Omega) = \frac{-j\Omega}{(a^2+\Omega^2)}$$

$$|X(j\Omega)| = \sqrt{X_R^2(j\Omega) + X_I^2(j\Omega)}$$

$$= \sqrt{\left(\frac{a}{(a^2+\Omega^2)}\right)^2 + \left(\frac{-\Omega}{(a^2+\Omega^2)}\right)^2} = \sqrt{\left(\frac{a^2+\Omega^2}{(a^2+\Omega^2)^2}\right)} = \sqrt{\left(\frac{1}{(a^2+\Omega^2)}\right)}$$

$$\angle X(j\Omega) = \tan^{-1}\left[\frac{X_I(j\Omega)}{X_R(j\Omega)}\right] = \tan^{-1}\left[\frac{-\Omega}{a}\right] = -\tan^{-1}\left[\frac{\Omega}{a}\right]$$

For a=2

Ω	**-5**	**-4**	**-3**	**-2**	**-1**	**0**	**1**	**2**	**3**	**4**	**5**
$\lvert X(j\Omega)\rvert$	0.185	0.22	0.277	0.35	0.44	0.5	0.44	0.35	0.277	0.22	0.185
$\angle X(j\Omega)$	68.19	63.43	56.30	45	26.56	0	26.56	45	56.30	63.43	68.19

1. Determine the inverse Fourier Transform

a. $X(j\Omega) = \begin{cases} 1 \ ; |\Omega| \le \omega \\ 0 \ ; \ Else \end{cases}$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} x(t)e^{j\Omega t}\,d\Omega$$

$$= \frac{1}{2\pi}\int_{-\omega}^{\omega} 1.e^{j\Omega t}\,d\Omega \quad = \frac{1}{2\pi}\left[\frac{e^{j\Omega t}}{jt}\right]_{-\omega}^{\omega}$$

$$= \frac{1}{\pi t}\left[\frac{e^{j\omega t} - e^{-j\omega t}}{2j}\right]$$

$$= \frac{1}{\pi t}\sin\omega t = \frac{\omega}{\omega\pi t}\sin\omega t = \frac{\omega}{\pi}\sin c\omega t$$

b. $X(j\Omega) = \dfrac{3(j\Omega)+14}{(j\Omega)^2 + 7(j\Omega)+12}$

$$= \frac{3(j\Omega)+14}{(j\Omega+3)(j\Omega+4)} = \frac{A}{(j\Omega+3)} + \frac{B}{(j\Omega+4)}$$

$$= \frac{A(j\Omega+4)+B(j\Omega+3)}{(j\Omega+3)(j\Omega+4)}$$

$$\Rightarrow 3(j\Omega)+14 = A(j\Omega+4)+B(j\Omega+3)$$

$$Put \ \ j\Omega = -3 \Rightarrow 3(-3)+14 = A(-3+4)+B(-3+3) \Rightarrow 5 = A$$

$$Put \ \ j\Omega = -4 \Rightarrow 3(-4)+14 = A(-4+4)+B(-4+3) \Rightarrow 2 = -B$$

$X(j\Omega) = \dfrac{5}{(j\Omega+3)} + \dfrac{-2}{(j\Omega+4)}$ [By taking Inverse Fourier Transform]

$$x(t) = 5e^{-3t}u(t) - 2e^{-4t}u(t)$$

Fourier Series

It is a mathematical tool which is used to analyze the periodic signals. There are two types

i) Exponential Fourier Series

ii) Trigonometric Fourier Series

Exponential Fourier Series

$$x(t) = \sum_{n-\infty}^{\infty} C_n e^{jn\Omega_o t}$$

$$Where\ C_o = \frac{1}{T}\int_0^T x(t)dt$$

$$C_n = \frac{1}{T}\int_0^T x(t)e^{-jn\Omega_o t}dt$$

Trigonometric Fourier Series

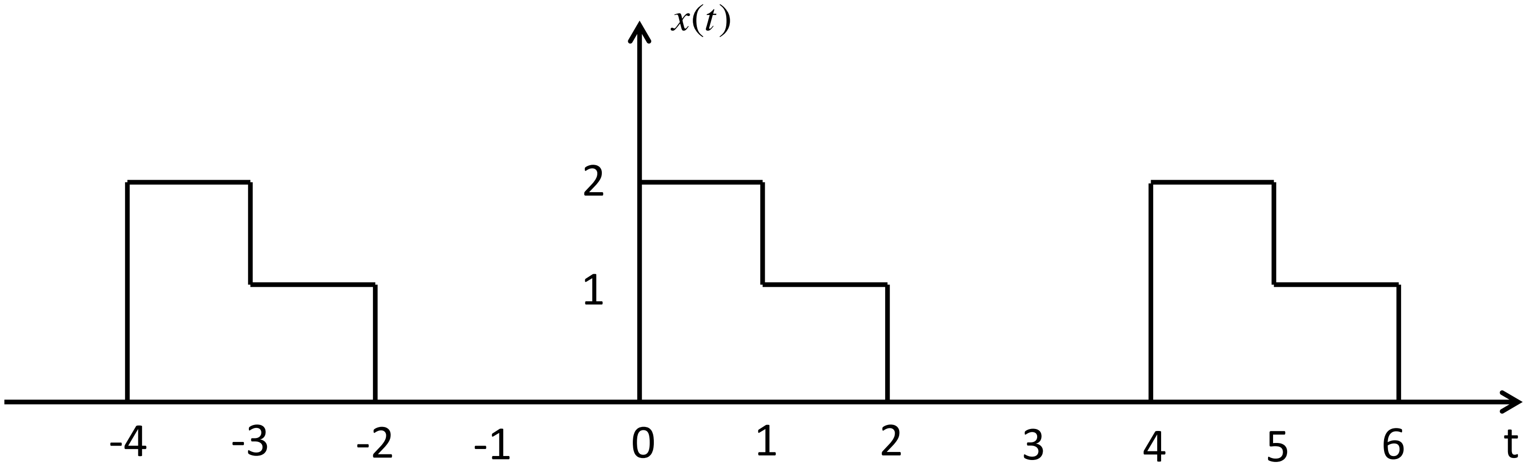
$$x(t) = \frac{a_o}{2} + \sum_{n=1}^{\infty} a_n \cos n\Omega_o t + \sum_{n=1}^{\infty} b_n \sin n\Omega_o t$$

$$Where\quad a_o = \frac{2}{T}\int_0^T x(t)dt$$

$$a_n = \frac{2}{T}\int_0^T x(t)\cos n\Omega_o t dt$$

$$b_n = \frac{2}{T}\int_0^T x(t)\sin n\Omega_o t dt$$

1. Compute the exponential Fourier series for the following signal



$$T = 4 \quad \Omega_0 = 2\pi / T = 2\pi / 4 = \pi / 2$$

$$C_0 = \tfrac{1}{T}\int_0^T x(t)dt = \frac{1}{4}\int_0^4 x(t)dt = \frac{1}{4}\left[\int_0^1 2dt + \int_1^2 dt + \int_2^4 0.dt\right]$$

$$= \frac{1}{4}\left[\int_0^1 2dt + \int_1^2 dt\right] = \frac{1}{4}[2+1]$$

$$C_0 = \frac{3}{4}$$

$$C_n = \frac{1}{T}\int_0^T x(t)e^{-jn\Omega_0 T}\ dt$$

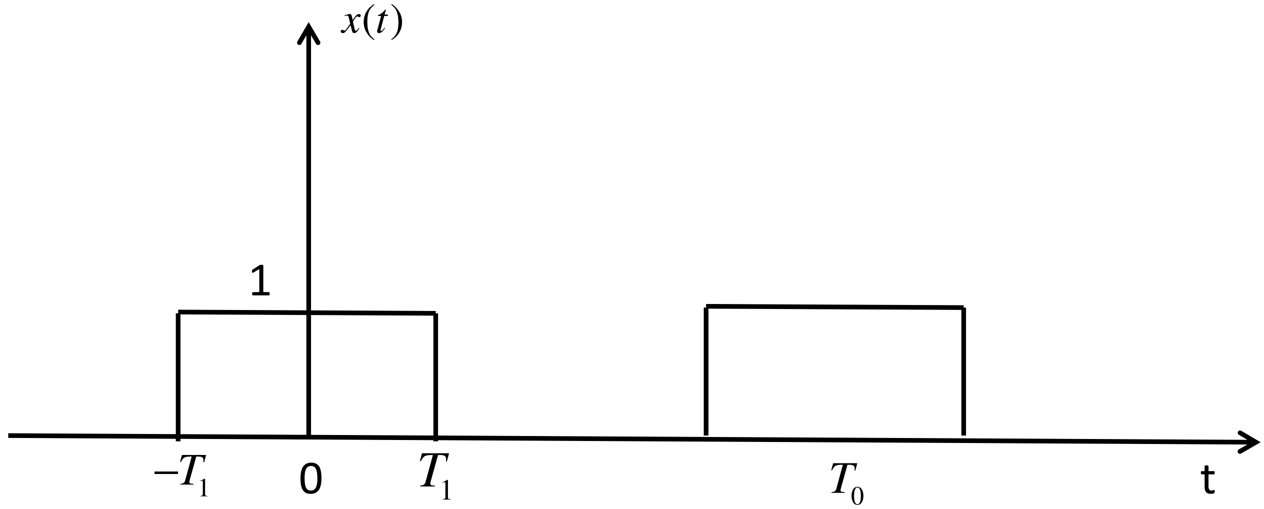
$$= \frac{1}{4}\int_0^4 x(t)e^{-j\frac{n\pi t}{2}}\ dt = \frac{1}{4}[\int_0^1 2\,e^{-j\frac{n\pi t}{2}}\ dt + \int_1^2 e^{-j\frac{n\pi t}{2}}\ dt + 0]$$

$$= \frac{1}{4}\left[2\left[e\frac{-j\frac{n\pi t}{2}}{-j\frac{n\pi t}{2}}\right]_0^1 + \left[e\frac{-j\frac{n\pi t}{2}}{-j\frac{n\pi t}{2}}\right]_1^2\right] = \frac{2}{4\,jn\pi}\ \left[2\left(e^{-jn\pi/2} - 1\right) + \left(e^{-jn\pi} - e^{-jn\pi/2}\right)\right]$$

$$= \frac{j}{2n\pi} \left[\left(2e^{-jn\pi/2} - 2\right) + \left(e^{-jn\pi} - e^{-jn\pi/2}\right) \right] = \frac{j}{n\pi}\left[\frac{1}{2}\, e^{-jn\pi/2} + \frac{1}{2}\left(-1\right)^n - 1 \right]$$

$$C_n = \frac{1}{jn\pi}\left[1 - \frac{1}{2}\left[e^{-jn\pi/2} - \left(-1\right)^n \right] \right]$$

2. Determine the exponential Fourier Series



$$T = T_0 \quad \Omega_0 = \frac{2\pi}{T_0}$$

$$C_0 = \tfrac{1}{T} \int_0^T x(t)dt = \frac{1}{T_0} \int_{-T_1}^{T_1} dt = \frac{1}{T_0}\left[T_1 + T_1 \right] = \frac{2T_1}{T_0}$$

$$C_n = \frac{1}{T_0} \int_{-T_1}^{T_1} x(t) e^{-j\Omega_0 t}\, dt = \frac{1}{T_0} \int_{-T_1}^{T_1} x(t) e^{\frac{-j2\pi nt}{T_0}}\, dt = \frac{1}{T_0} \left[e^{\frac{-j2\pi nt}{T_0}} \over \frac{-j2\pi nt}{T_0} \right]_{-T_1}^{T_1}$$

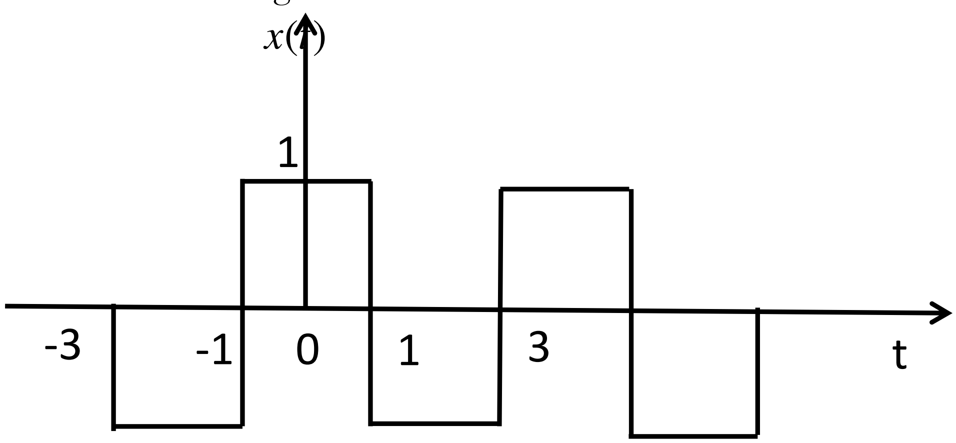
$$= \frac{T_0}{T_0 j2\pi n}\left[e^{-j2\pi n T_{1/T_0}} - e^{j2\pi n T_{1/T_0}} \right] = \tfrac{1}{j2\pi n}\left[e^{j2\pi n T_{1/T_0}} - e^{-j2\pi n T_{1/T_0}} \right]$$

$$= \tfrac{1}{n\pi}\left[\frac{e^{j2\pi n T_{1/T_0}} - e^{-j2\pi n T_{1/T_0}}}{2j} \right] = \tfrac{1}{n\pi}\left[\sin\left(\frac{2\pi n T_1}{T_0} \right) \right]$$

$$C_n = \tfrac{1}{n\pi}\left[\sin\left(\frac{2\pi n T_1}{T_0}\right)\right]$$

Trigonometric Fourier series

1. Determine the Trigonometric Fourier Series



$$\Omega_0 = \frac{2\pi}{T} \qquad T = 4 \qquad \Omega_0 = \frac{2\pi}{4} = \frac{\pi}{2} \qquad x(t) = \begin{cases} 1 & -1 \le t \le 1 \\ -1 & 1 \le t \le 3 \end{cases}$$

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos n\Omega_0 t \; + b_n \sin n\Omega_0 t\right]$$

$$x(t) = a_0 \; + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi t}{2}\right) + b_n \sin\left(\frac{n\pi t}{2}\right)\right]$$

$$a_0 = \tfrac{1}{T}\int_1^3 x(t)dt = \frac{1}{4}\left[\int_{-1}^{1} dt + \int_1^3 -1dt\right] = \frac{1}{4}\left[(1+1) - 1(3-1)\right]$$

$$a_0 = \frac{1}{4}(2-2) = 0$$

$$a_n = \tfrac{2}{T}\int_1^3 x(t)\cos\left(\frac{n\pi t}{2}\right)dt$$

$$=\frac{1}{2}\left[\int_{-1}^{1}\cos\left(\frac{n\pi t}{2}\right)dt-\int_{1}^{3}\cos\left(\frac{n\pi t}{2}\right)dt\quad\right]$$

$$\frac{1}{2}\left[\left[\frac{2}{n\pi}\sin\left(\frac{n\pi t}{2}\right)\right]_{-1}^{1}-\left[\frac{2}{n\pi}\sin\left(\frac{n\pi t}{2}\right)\right]_{1}^{3}\quad\right]$$

$$=\frac{1}{n\pi}\left[\sin\left(\frac{n\pi}{2}\right)+\sin\left(\frac{n\pi}{2}\right)-\left(\sin\left(\frac{3n\pi}{2}\right)-\sin\left(\frac{n\pi}{2}\right)\right)\right]$$

$$a_n=\frac{4}{n\pi}\sin\left(\tfrac{n\pi}{2}\right)$$

$$b_n=\tfrac{2}{T}\int_{-1}^{3}x(t)\sin\left(\frac{n\pi t}{2}\right)dt$$

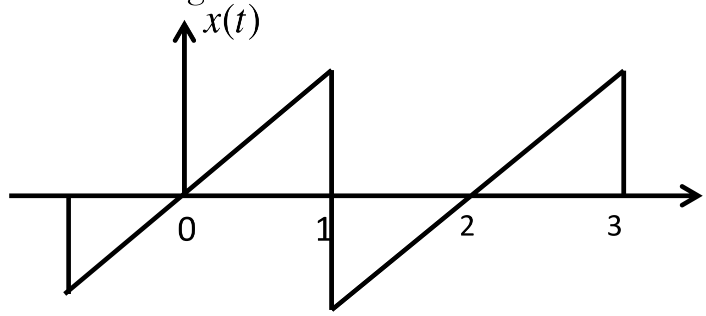
$$=\frac{1}{2}\left[\int_{-1}^{1}\sin\left(\frac{n\pi t}{2}\right)dt+\int_{1}^{3}\sin\left(\frac{n\pi t}{2}\right)dt\quad\right]$$

$$=\frac{2}{2n\pi}\left[-\cos\left(\frac{n\pi}{2}\right)+\cos\left(\frac{n\pi}{2}\right)+\left(-\cos\left(\frac{n\pi t}{2}\right)\right)_{1}^{3}\quad\right]$$

$$=\frac{1}{n\pi}\left[0-\cos\left(\frac{3n\pi}{2}\right)+\cos\left(\frac{n\pi}{2}\right)\right]=0$$

$$x(t)=\sum_{n=1}^{\infty}a_n\cos\left(\frac{n\pi t}{2}\right)=\sum_{n=1}^{\infty}\frac{4}{n\pi}\sin\left(\frac{n\pi}{2}\right)\cos\left(\frac{n\pi t}{2}\right)$$

2. Determine the Trigonometric Fourier Series



$T=2$

$\Omega_0=\frac{2\pi}{T}=\pi$

$$x(t)=\begin{cases} t & -1\le t\le 1 \\ 0 & else\end{cases}$$

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(n\Omega_0 t\right) + b_n \sin\left(n\Omega_0 t\right) \right]$$

$$a_0 = \tfrac{1}{T} \int x(t)dt = \frac{1}{2} \int_{-1}^{1} t dt = \frac{1}{2} \left(\frac{t^2}{2} \right)_{-1}^{1} = \frac{1}{4}(1-1) = 0$$

$$a_n = \tfrac{2}{T} \int_{-1}^{1} t \cos\left(n\pi t\right) dt$$

$$u = t \qquad v = \cos n\pi t$$

$$u' = 1 \qquad v_1 = \sin \frac{n\pi t}{n\pi}$$

$$u'' = 0 \qquad v_2 = -\cos \tfrac{n\pi t}{n^2\pi^2}$$

$$= \frac{2}{2} \left[\frac{t}{n\pi} \sin n\pi t + \cos \tfrac{n\pi t}{n^2\pi^2} \right] \quad a_n = 0$$

$$b_n = \tfrac{2}{T} \int_{-1}^{1} t \sin\left(n\pi t\right) dt$$

$$u = t \qquad v = \sin n\pi t$$

$$u' = 1 \qquad v_1 = -\cos \frac{n\pi t}{n\pi}$$

$$u'' = 0 \qquad v_2 = -\sin \tfrac{n\pi t}{n^2\pi^2}$$

$$= \left[-\frac{t}{n\pi} \cos n\pi t + \sin \tfrac{n\pi t}{n^2\pi^2} \right]_{-1}^{1}$$

$$= -\frac{\cos n\pi}{n\pi} + \frac{\sin n\pi}{n^2\pi^2} - \left(\frac{\cos n\pi}{n\pi} - \frac{\sin n\pi}{n^2\pi^2} \right)$$

$$b_n = \frac{-2}{n\pi} \cos n\pi = \frac{-2}{n\pi} \left[(-1)^n \right] = \frac{-2}{\pi} \left[\frac{(-1)^n}{n} \right]$$

$$x(t) = \sum_{n=1}^{\infty} \frac{-2}{\pi} \left[\frac{(-1)^n}{n} \right] \sin n\pi t$$

UNIT-3
Analysis of Continuous Time Signals

Syllabus:

Impulse Response-Convolution Integrals, Differential Equation, Fourier & Laplace Transform in Analysis of Continuous Time Systems; System Connected in Series & Parallel Combinations

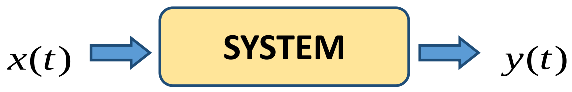
Session	Session Learning	Topic
1	1	Continuous Time Systems & its Representation
	2	LTI Systems and Transfer function
2	1	Differential Equation
	2	Analysis of LTI Systems
3	1	Analysis of Response
	2	Analysis of Step Response
4	1	Analysis of Impulse Response
	2	Convolutional Integral
5	1	Properties of Convolutional Integral
	2	Convolutional Integral using formula
6	1	Convolutional Integral using Graph
	2	Convolutional Integral using Graph
7	1	Systems Connected in Series
	2	Systems Connected in Parallel
8	1	Basic Elements of Block Diagram Representation
	2	Types of Block Diagram Representation
9	1	Direct form-I Representation & Realization
	2	Direct form-II Representation & realization
10	1	Cascade representation
	2	Realization using Cascade Representation
11	1	Parallel Representation
	2	Realization using Parallel representation
12	1	Frequency Response Analysis
	2	Stability Analysis

Continuous Time Signal:

It is a signal in which the amplitude can be measured at any time instant. It is denoted by x(t).

Continuous Time System:

The system which is used to analysis the continuous time signal. It is denoted by the differential equation.



$$y(t) = T\left[x(t)\right]$$

Transfer Function:

It is defined as the ratio of Laplace transformation of output signal to the Laplace transformation of input signal. It is denoted by H(s). $H(s) = \frac{L\left[y(t)\right]}{L\left[x(t)\right]} = \frac{Y(s)}{X(s)}$

Response:

It is defined as the output of the system, it is denoted by *y(t)*. It is classified into two types, they are of

i) **Step Response** :: When the input is unit step signal, then the output is called as Step response.

$$H(s) = \frac{Y(s)}{X(s)}$$

$$Y(s) = H(s)X(s)$$

$$L^{-1}\left[Y(s)\right] = L^{-1}\left[H(s)X(s)\right] \qquad X(s) = \frac{1}{s}$$

ii) **Impulse Response** :: When the input is unit impulse signal, then the output is called as Impulse response.

$$H(s) = \frac{Y(s)}{X(s)}$$

$$Y(s) = H(s)X(s)$$

$$L^{-1}\left[Y(s)\right] = L^{-1}\left[H(s)X(s)\right] \qquad X(s) = 1$$

$$L^{-1}\left[Y(s)\right] = L^{-1}\left[H(s)\right]$$

$$y(t) = h(t)$$

Frequency Response:

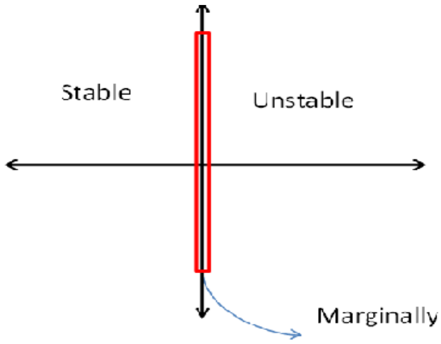
It is representation, in which frequency domain analysis is done using the Fourier Transform

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)}$$

Stability:

It is a factor which is used to analysis the stability of the system. It is justified mathematically and graphically based system representation.

- If the Transfer function H(s) is given, determine the poles and zeros, plot the graph
    - If the poles and zeros lies only in the left hand side of the axis then the given system is Stable System.
    - If the poles and zeros lies in right hand side of the axis then the given system is Unstable System.
    - If the poles and zeros lies in the margin, then the system is Marginally Stable System.



- If the Impulse Response h(t) is given, the stability is verified by the formula. $\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$
- If the response y(t) of the system is given in terms of x(t) ; Determine the impulse response h(t) by substituting $\delta(t)$ in x(t) and followed by using the above formula.

Check whether the given system is stable or unstable system

1. $h(t) = \frac{1}{Rc} e^{-t/RC} u(t)$

$$\int_{-\infty}^{\infty} |h(t)|\, dt = \int_{-\infty}^{\infty} \left| \frac{1}{Rc} e^{-t/RC} u(t) \right| dt = \frac{1}{Rc} \int_{0}^{\infty} \left| e^{-t/RC} \right| dt = \frac{1}{Rc} \left[\frac{e^{-t/RC}}{-1/RC} \right]_0^{\infty} = \frac{1}{Rc} [0 + RC] = 1$$

The given system is stable

2. $h(t) = e^{2t} u(t-1)$

$$\int_{-\infty}^{\infty} |h(t)|\, dt = \int_{-\infty}^{\infty} \left| e^{2t} u(t-1) \right| dt = \int_{1}^{\infty} \left| e^{2t} \right| dt = \left[\frac{e^{2t}}{2} \right]_0^{\infty} = \infty$$

The given system is unstable

3. $H(s) = \frac{1}{(s+1)(s+2)}$

The poles are s=-1, s=-2; It lies in the left hand side of the axis, so the system is stable.

Causality:

$$h(t) = 0; t < 0$$

Basic Formulae

	With Initial Conditions	Without Initial Conditions
$L\left[\frac{dx(t)}{dt}\right]$	$sX(s) - x(0)$	$sX(s)$
$L\left[\frac{d^2x(t)}{dt^2}\right]$	$s^2X(s) - sx(0) - x(1)$	$s^2X(s)$
$L\left[\frac{d^3x(t)}{dt^3}\right]$	$s^3X(s) - s^2x(0) - sx(1) - x(2)$	$s^3X(s)$

1. **The input – output relation of a system at initial rest is given by** $\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = \frac{dx(t)}{dt} + 2x(t)$ **using the Laplace Transform. Find the system transfer function, frequency response, impulse response**

Given: $\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = \frac{dx(t)}{dt} + 2x(t)$ **Initially at rest**

Taking Laplace Transform on both sides

$$L\left[\frac{d^2y(t)}{dt^2}\right] + L\left[4\frac{dy(t)}{dt}\right] + L\left[3y(t)\right] = L\left[\frac{dx(t)}{dt}\right] + L\left[2x(t)\right]$$

$$s^2Y(s) + 4sY(s) + 3Y(s) = sX(s) + 2X(s)$$

$$Y(s)\left[s^2 + 4s + 3\right] = X(s)\left[s + 2\right]$$

i) **Transfer Function** :: $H(s) = \frac{Y(s)}{X(s)} = \frac{s+2}{s^2 + 4s + 3}$

ii) **Frequency Response** :: $s \to j\omega$; $H(j\omega) = \frac{j\omega + 2}{(j\omega)^2 + 4j\omega + 3}$

iii) **Impulse Response**

$$H(s) = \frac{s+2}{s^2 + 4s + 3} = \frac{s+2}{(s+1)(s+3)} \qquad \text{--------1}$$

$$H(s) = \frac{A}{(s+1)} + \frac{B}{(s+3)} = \frac{A(s+3) + B(s+1)}{(s+1)(s+3)} \qquad \text{--------2}$$

Equating the equations 1 & 2 we obtain

$$s + 2 = A(s+3) + B(s+1)$$

To determine A and B

Put $s = -3$ :: $-3 + 2 = A(-3+3) + B(-3+1) \Rightarrow -1 = A(0) + B(-2) \Rightarrow B = \frac{1}{2}$

Put $s = -1$ :: $-1 + 2 = A(-1+3) + B(-1+1) \Rightarrow 1 = A(2) + B(0) \Rightarrow A = \frac{1}{2}$

$$H(s)=\frac{1/2}{(s+1)}+\frac{1/2}{(s+3)}$$

Taking Inverse Laplace Transform $h(t)=L^{-1}[H(s)]=L^{-1}\left[\frac{1/2}{(s+1)}\right]+L^{-1}\left[\frac{1/2}{(s+3)}\right]$

$$h(t)=\frac{1}{2}\left[e^{-t}+e^{-3t}\right]u(t)$$

Problem for Practice:

1. **Consider an LTI system whose response to the input** $\left(e^{-t}+e^{-3t}\right)u(t)$ **is** $\left(2e^{-t}-2e^{-4t}\right)u(t)$**. Find the system impulse response.**
2. **For a continuous time LTI system defined by its impulse response** $h(t)=e^{-2t}u(t)$**. Determine its transfer function , unit step response and response for the input** $x(t)=e^{-t}u(t)$**.**
3. **For an LTI system defined by** $\frac{d^2y(t)}{dt^2}-4\frac{dy(t)}{dt}+5y(t)=5x(t)$**. Find the response of the system y(t) for an input of x(t)=u(t) with an initial conditions of** $y(0)=1;\left.\frac{dy(t)}{dt}\right|_{t=0}=2$
4. **Using Laplace Transform find the output y(t) of an LTI system represented by** $\frac{d^2y(t)}{dt^2}+3\frac{dy(t)}{dt}+2y(t)=x(t)$ **for an input** $x(t)=\delta(t)$ **assume that the system is initially at rest.**

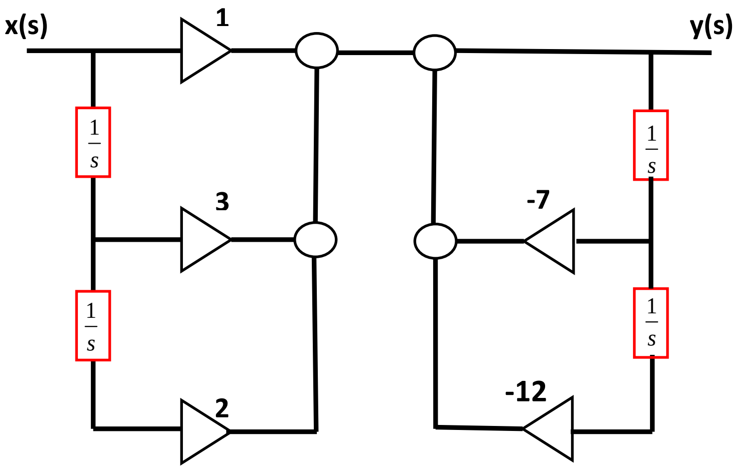
Block Diagram Representation:

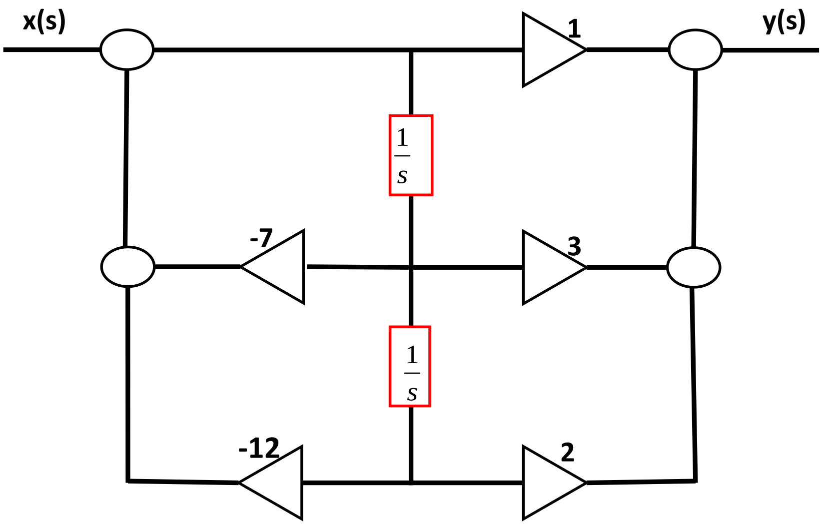
1. **Realize the given transfer function in terms of direct form-I, direct form-II, cascade and parallel** $H(s)=\frac{s^2+3s+2}{s^2+7s+12}$**. [The same problem is also given in the form of differential equation** $\frac{d^2y(t)}{dt^2}+7\frac{dy(t)}{dt}+12y(t)=\frac{d^2x(t)}{dt^2}+3\frac{dx(t)}{dt}+2x(t)$ **]**

i) Realization of Direct form-I & Direct Form-II

$$H(s)=\frac{s^2+3s+2}{s^2+7s+12}$$ **[Divide numerator & denominator by s²]**

$$H(s)=\frac{\left(s^2+3s+2\right)/s^2}{\left(s^2+7s+12\right)/s^2}=\frac{1+\frac{3}{s}+\frac{2}{s^2}}{1+\frac{7}{s}+\frac{12}{s^2}}=\frac{1+\frac{3}{s}+\frac{2}{s^2}}{1-\left(-\frac{7}{s}-\frac{12}{s^2}\right)}$$

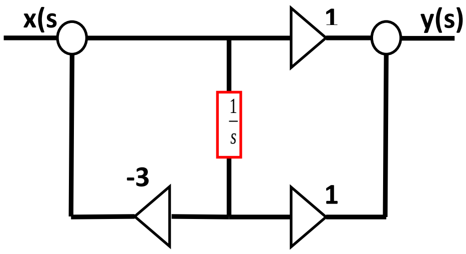




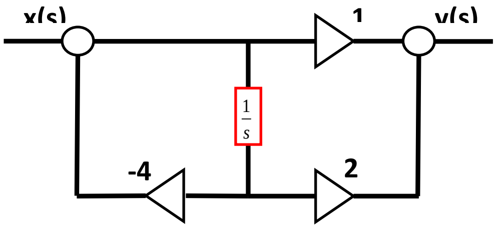
ii) **Realization of Cascade form:**

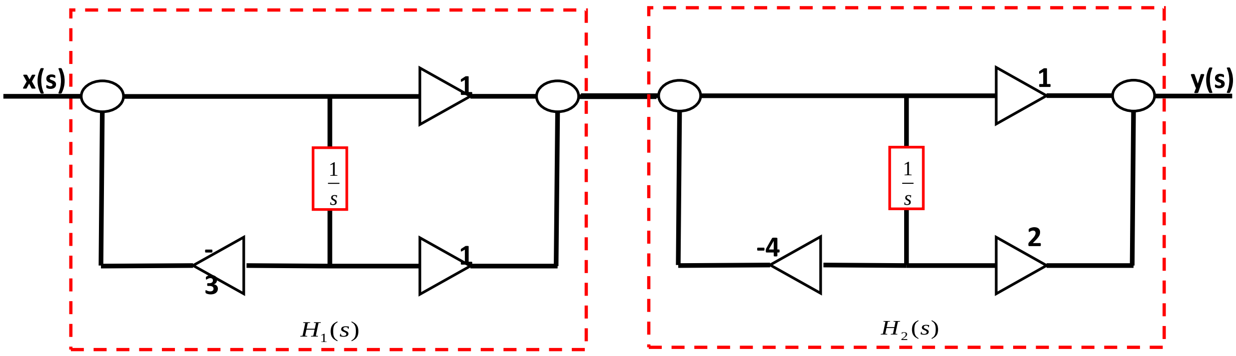
$$H(s)=\frac{s^2+3s+2}{s^2+7s+12}=\frac{(s+1)(s+2)}{(s+3)(s+4)}=\frac{(s+1)}{(s+3)}\bullet\frac{(s+2)}{(s+4)}$$

$$H_1(s)=\frac{(s+1)}{(s+3)}=\frac{\left(1+\frac{1}{s}\right)}{\left(1+\frac{3}{s}\right)}=\frac{\left(1+\frac{1}{s}\right)}{\left(1-\left(-\frac{3}{s}\right)\right)}$$



$$H_2(s)=\frac{(s+2)}{(s+4)}=\frac{\left(1+\frac{2}{s}\right)}{\left(1+\frac{4}{s}\right)}=\frac{\left(1+\frac{2}{s}\right)}{\left(1-\left(-\frac{4}{s}\right)\right)}$$





iii) Realization of Parallel form:

$$H(s)=\frac{s^2+3s+2}{s^2+7s+12}=\frac{s^2+3s+2}{(s+3)(s+4)}=\frac{A}{(s+3)}+\frac{B}{(s+4)} \qquad \text{---------1}$$

$$H(s)=\frac{A(s+4)+B(s+3)}{(s+3)(s+4)} \qquad \text{---------2}$$

Equating the equations 1 & 2

$$s^2+3s+2=A(s+4)+B(s+3)$$

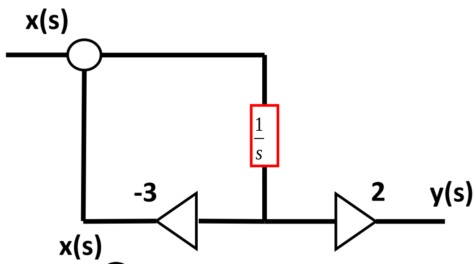
To Determine A & B

$$s=-4\Rightarrow(-4)^2+3(-4)+2=A(-4+4)+B(-4+3)\Rightarrow 16-12+2=-B\Rightarrow B=6$$

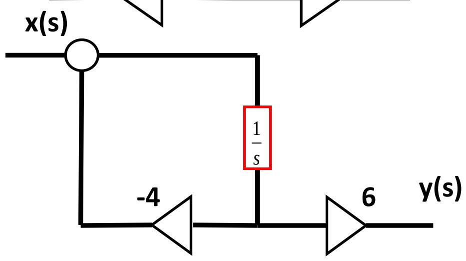
$$s=-3\Rightarrow(-3)^2+3(-3)+2=A(-3+4)+B(-3+3)\Rightarrow 9-9+2=A\Rightarrow A=2$$

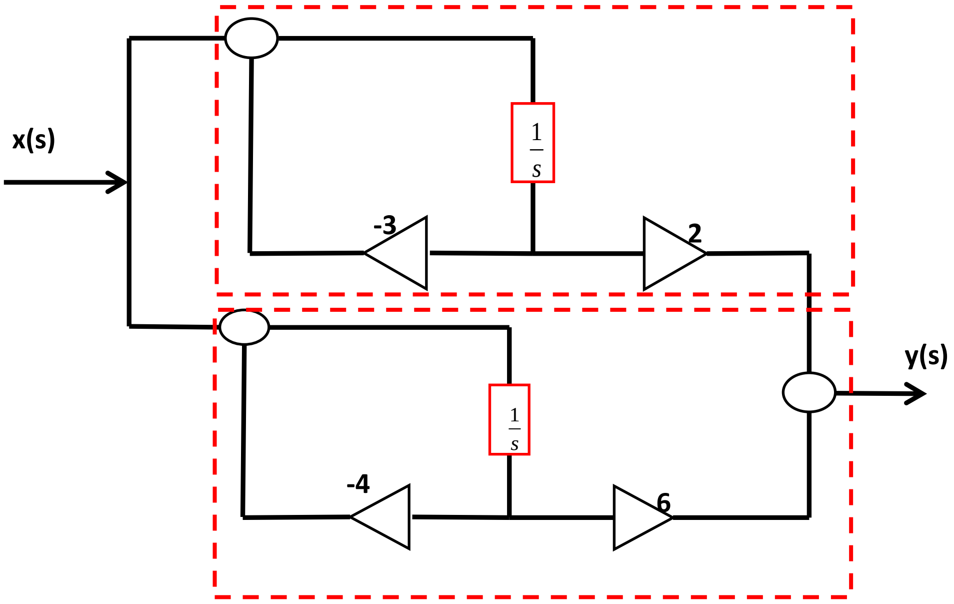
$$H(s)=\frac{2}{(s+3)}+\frac{6}{(s+4)}$$

$$H_1(s)=\frac{2}{(s+3)}=\frac{\frac{2}{s}}{\left(1+\frac{3}{s}\right)}=\frac{\frac{2}{s}}{\left(1-\left(-\frac{3}{s}\right)\right)}$$



$$H_2(s)=\frac{6}{(s+4)}=\frac{\frac{6}{s}}{\left(1+\frac{4}{s}\right)}=\frac{\frac{6}{s}}{\left(1-\left(-\frac{4}{s}\right)\right)}$$





Problem for Practice:

1. **Realize in direct form-II**

   a. $H(s)=\dfrac{s^3+2s+3}{2s^3+3s^2+0.5s+1}$

   b. $H(s)=\dfrac{s^2+2s+3}{s^2+4s+7}$

   c. $H(s)=\dfrac{s+1}{s+2}$

   d. $H(s)=\dfrac{1}{s+1}$

2. **Realize in Cascade & Parallel form**

   a. $H(s)=\dfrac{s^2+3s+2}{s^2+7s+12}$

   b. $H(s)=\dfrac{1}{s^2+3s+2}$

   c. $H(s)=\dfrac{(s+2)(s-5)}{(s+4)(s^2+s+3)}$

   d. $H(s)=\dfrac{s^2+2s}{(s+6)(s^2-7s+10)}$

Convolutional Integral:

It is technique which is used to determine the output of the system.

$$y(t)=x(t)*h(t)=\int_{-\infty}^{\infty}x(\tau)h(t-\tau)d\tau$$

Properties of Convolutional Integral

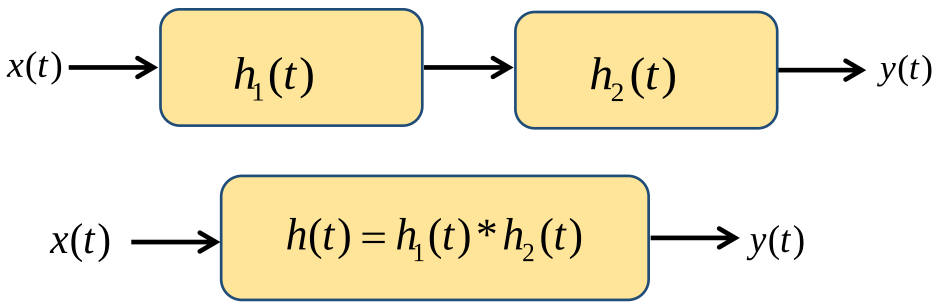
i) Commutative : $x(t)*h(t)=h(t)*x(t)$

ii) Distributive : $x(t)*[h_1(t)+h_2(t)]=x(t)*h_1(t)+x(t)*h_2(t)$

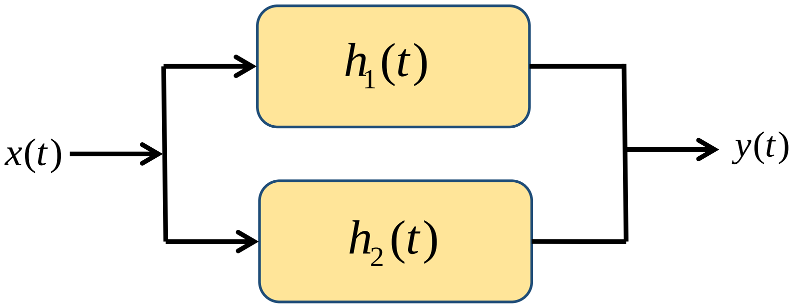
iii) Associative : $x(t)*[h_1(t)*h_2(t)]=[x(t)*h_1(t)]*h_2(t)$

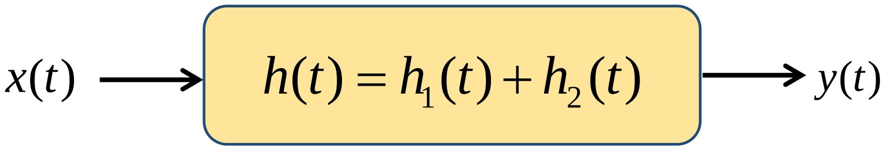
Interconnection of Impulse Response

1. Series Combination



2. Parallel Combination





1. Find the convolution Integral for the following signal.

   a. $x_1(t) = e^{-at}u(t)$ & $x_2(t) = e^{-bt}u(t)$

$$y(t) = \int_0^{\tau} x(\tau)h(t-\tau)d\tau$$

$$y(t) = \int_0^{\tau} e^{-a\tau}u(\tau)e^{-b(t-\tau)}u(t-\tau)d\tau$$

$$y(t) = \int_0^{\tau} e^{-a\tau}e^{-b(t-\tau)}u(\tau)u(t-\tau)d\tau$$

$$y(t) = \int_0^{t} e^{-a\tau}e^{-b(t-\tau)}d\tau = \int_0^{t} e^{-a\tau}e^{-bt}e^{b\tau}d\tau$$

$$= e^{-bt}\int_0^{t} e^{-a\tau}e^{b\tau}d\tau = e^{-bt}\int_0^{t} e^{(b-a)\tau}d\tau = e^{-bt}\left[\frac{e^{(b-a)\tau}}{(b-a)}\right]_0^t$$

$$= \frac{e^{-bt}}{(b-a)}\left[e^{(b-a)\tau}\right]_0^t = \frac{e^{-bt}}{(b-a)}\left[e^{(b-a)t} - 1\right] = \frac{e^{-bt}}{(b-a)}\left[e^{-at}e^{bt} - 1\right]$$

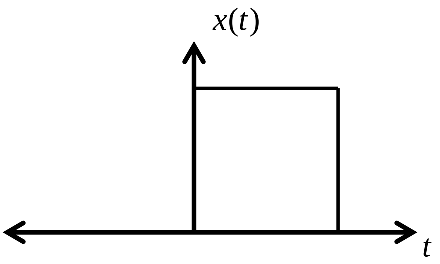
$$y(t) = \frac{1}{(b-a)}\left[e^{-at} - e^{-bt}\right]u(t)$$

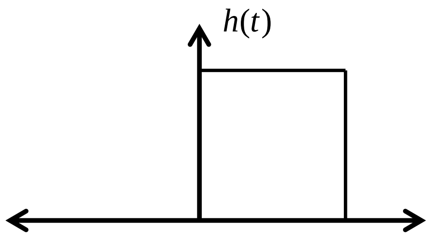
2. $x_1(t) = \sin t u(t)$ & $x_2(t) = u(t)$

$$y(t) = \int_0^{\tau} x(\tau)h(t-\tau)d\tau$$

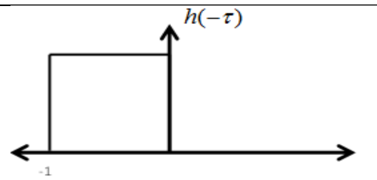
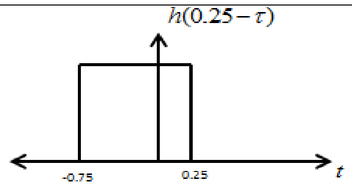
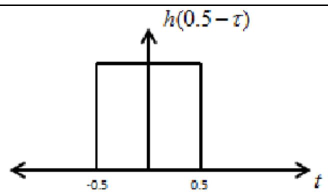
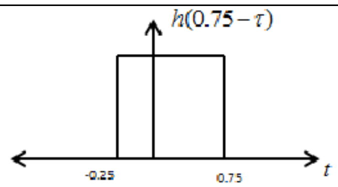
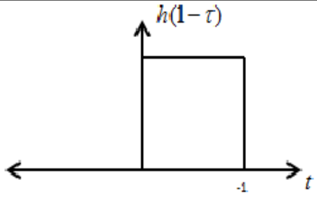
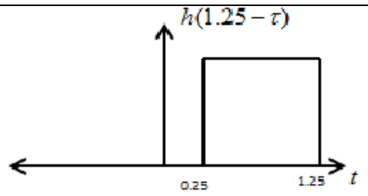
$$y(t) = \int_0^{\tau} \sin\tau u(\tau)u(t-\tau)d\tau = \int_0^{t} \sin\tau d\tau = \left[-\cos\tau\right]_0^t = -\cos t + 1 = [1-\cos t]u(t)$$

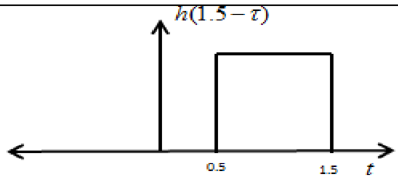
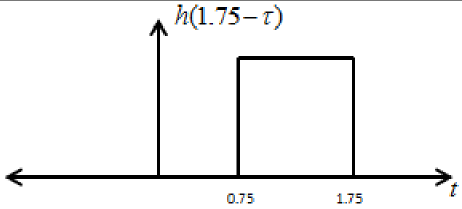
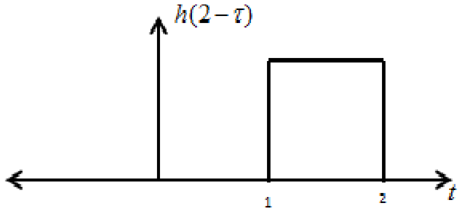
3. Determine the Convolutional Integral





$$y(t)=\int_{0}^{\tau} x(\tau)h(t-\tau)d\tau$$

t=0	$y(t)=\int_{-1}^{1} x(\tau)h(1-\tau)d\tau$ $y(1)=0$	
t=0.25	$y(t)=\int_{0}^{0.25} x(\tau)h(0.25-\tau)d\tau$ $y(0.25)=0.25-0$ $y(0.25)=0.25$	
t=0.5	$y(t)=\int_{0}^{0.5} x(\tau)h(0.5-\tau)d\tau$ $y(0.5)=0.5-0$ $y(0.5)=0.5$	
t=0.75	$y(t)=\int_{0}^{0.75} x(\tau)h(0.75-\tau)d\tau$ $y(0.75)=0.75-0$ $y(0.75)=0.75$	
t=1	$y(t)=\int_{0}^{1} x(\tau)h(1-\tau)d\tau$ $y(1)=1-0$ $y(1)=1$	
t=1.25	$y(t)=\int_{0.25}^{1.25} x(\tau)h(1.25-\tau)d\tau$ $y(1.25)=1-0.25$ $y(1.25)=0.75$	

t=1.5	$y(t)=\int_{0.5}^{1.5} x(\tau)h(1.5-\tau)d\tau$ $y(1.5)=1-0.5$ $y(1.5)=0.5$	
t=1.75	$y(t)=\int_{0.75}^{1.75} x(\tau)h(1.75-\tau)d\tau$ $y(1.75)=1-0.75$ $y(1.75)=0.25$	
t=2	$y(t)=\int_{1}^{2} x(\tau)h(2-\tau)d\tau$ $y(2)=0$	

State Variable Matrix Representation

State equation: $Q^{\bullet} = AQ + BX$

Output equation: $Y = CQ + DX$

Determination of Transfer Function:

$Q^{\bullet} = AQ + BX$ **(Taking Laplace Transform on both sides)**

$$sQ(s) = AQ(s) + BX(s)$$
$$sQ(s) - AQ(s) = BX(s)$$
$$Q(s)(sI - A) = BX(s)$$
$$Q(s) = \frac{BX(s)}{(sI - A)}$$
$$Q(s) = (sI - A)^{-1} BX(s)$$

$Y = CQ + DX$ **(Taking Laplace Transform on both sides)**

$$Y(s) = CQ(s) + DX(s)$$
$$Y(s) = C(sI - A)^{-1} BX(s) + DX(s)$$
$$Y(s) = [C(sI - A)^{-1} B + D]X(s)$$
$$H(s) = \frac{Y(s)}{X(s)} = [C(sI - A)^{-1} B + D]$$

1. Determine the Transfer Function $\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x(t) \quad y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$

Given : $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$; $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$; $C = \begin{bmatrix} 0 & 1 \end{bmatrix}$; $D = 0$

$$H(s) = C[sI - A]^{-1} B + D$$

$$sI = s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix}$$

$$sI - A = \begin{bmatrix} s-2 & 1 \\ 1 & s-2 \end{bmatrix}$$

$$(sI - A)^{-1} = \frac{adj(sI - A)}{\det(sI - A)}$$

$$adj(sI - A) = \begin{bmatrix} s-2 & -1 \\ -1 & s-2 \end{bmatrix}$$

$$\det(sI - A) = (s-2)^2 - 1 = s^2 + 4 - 4s - 1 = s^2 - 4s + 3$$

$$(sI - A)^{-1} = \begin{bmatrix} \frac{s-2}{s^2-4s+3} & \frac{-1}{s^2-4s+3} \\ \frac{-1}{s^2-4s+3} & \frac{s-2}{s^2-4s+3} \end{bmatrix}$$

$$H(s) = C(sI - A)^{-1} B + D$$

$$H(s) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{s-2}{s^2-4s+3} & \frac{-1}{s^2-4s+3} \\ \frac{-1}{s^2-4s+3} & \frac{s-2}{s^2-4s+3} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + [0]$$

$$H(s) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{-1}{s^2-4s+3} \\ \frac{s-2}{s^2-4s+3} \end{bmatrix}$$

$$H(s) = \frac{s-2}{s^2-4s+3}$$

2. Determine the A,B,C,D $\frac{d^3y(t)}{dt^3}+3\frac{d^2y(t)}{dt^2}+5\frac{dy(t)}{dt}+6y(t)=\frac{d^2x(t)}{dt^2}+6\frac{dx(t)}{dt}+5x(t)$

Given : $\frac{d^3y(t)}{dt^3}+3\frac{d^2y(t)}{dt^2}+5\frac{dy(t)}{dt}+6y(t)=\frac{d^2x(t)}{dt^2}+6\frac{dx(t)}{dt}+5x(t)$

Taking Laplace Transform on both sides

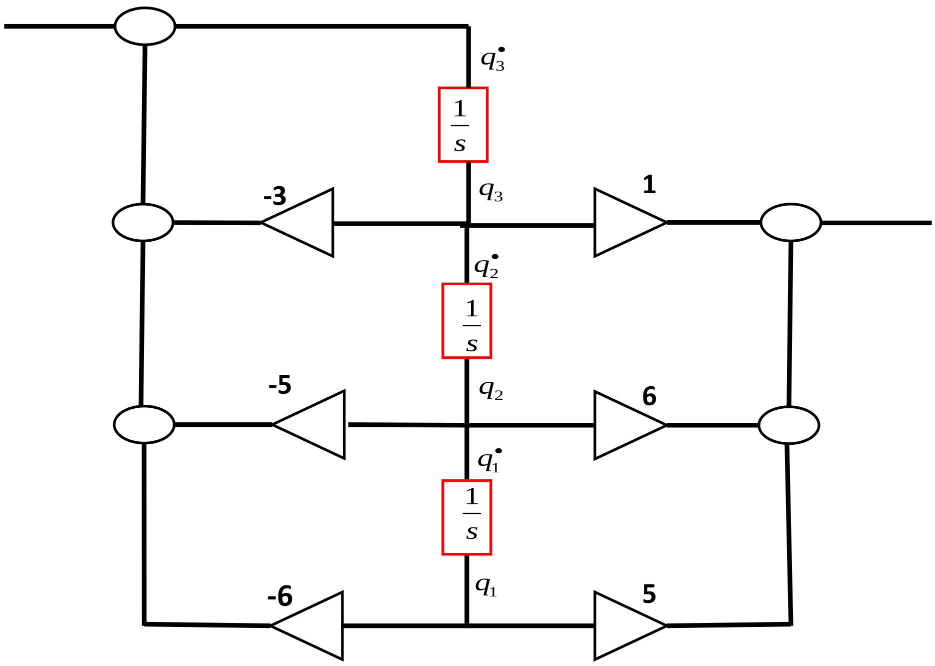
$$L\left[\frac{d^3y(t)}{dt^3}\right]+L\left[3\frac{d^2y(t)}{dt^2}\right]+L\left[5\frac{dy(t)}{dt}\right]+L\left[6y(t)\right]=L\left[\frac{d^2x(t)}{dt^2}\right]+L\left[6\frac{dx(t)}{dt}\right]+L\left[5x(t)\right]$$

$$s^3Y(s)+3s^2Y(s)+5sY(s)+6Y(s)=s^2X(s)+6sX(s)+5X(s)$$

$$[s^3+3s^2+5s+6]Y(s)=[s^2+6s+5]X(s)$$

$$H(s)=\frac{Y(s)}{X(s)}=\frac{[s^2+6s+5]}{[s^3+3s^2+5s+6]}$$

$$H(s)=\frac{\frac{1}{s}+\frac{6}{s^2}+\frac{5}{s^3}}{1+\frac{3}{s}+\frac{5}{s^2}+\frac{6}{s^3}}=\frac{\frac{1}{s}+\frac{6}{s^2}+\frac{5}{s^3}}{1-\left(-\frac{3}{s}-\frac{5}{s^2}-\frac{6}{s^3}\right)}$$



$$\dot{q}_1=q_2$$

$$\dot{q}_2=q_3$$

$$\dot{q}_3=-3q_3-5q_2-6q_1+x(t)$$

$$y(t)=q_3+6q_2+5q_1$$

$$\begin{bmatrix}\dot{q}_1\\ \dot{q}_2\\ \dot{q}_3\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -6 & -5 & -3\end{bmatrix}\begin{bmatrix}q_1\\ q_2\\ q_3\end{bmatrix}+\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}x(t) \qquad y=\begin{bmatrix}5 & 6 & 1\end{bmatrix}\begin{bmatrix}q_1\\ q_2\\ q_3\end{bmatrix}+[0]x(t)$$

$$A=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -6 & -5 & -3\end{bmatrix} \qquad B=\begin{bmatrix}0\\ 0\\ 1\end{bmatrix} \qquad C=\begin{bmatrix}5 & 6 & 1\end{bmatrix} \qquad D=[0]$$